Computing Real Witness Points of Positive Dimensional Polynomial Systems

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Abstract

We consider a critical point method for finding certain solution (witness) points on real solution components of real polynomial systems of equations. The method finds points that are critical points of the distance from a plane to the component with the requirement that certain regularity conditions are satisfied. In this paper we analyze the numerical stability and complexity of the method. We aim to find at least one well conditioned witness point on each connected component by using perturbation, path tracking and projection techniques. An optimal-direction strategy and an adaptive step size control strategy for path following on high dimensional components are given.

Categories and Subject Descriptors: G.1.8 **General Terms:** algorithms, design

Keywords: numerical algebraic geometry, real algebraic geometry, homotopy continuation, singular critical points, witness points, differential algebraic equations.

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1 Introduction

This paper is a contribution to the development of numerical algorithms for computational algebraic geometry pioneered by Sommese, Wampler, Verschelde and others [34, 7]. Recently this direction was extended to the real case by Lu [25], Besana et al. [10], and Hauenstein [18]. Important early work on exactly obtaining real points on components is given by Rouillier et al [27] and Safey El Din et al [29, 30].

We are aiming to find at least one point in each connected component of a polynomial system. A widely used method is the so-called "critical point" method. In symbolic computation, the study of a smooth and compact positive dimensional real variety by choosing a projection to reduce to zero dimensional critical locus can be found in [2]. It is related to the work by Bank et al. [6, 5] on polar varieties of complete intersections. To remove the compactness assumption, the authors introduced a distance function to minimize the distance to a given point in [27, 1]. A significant improvement due to Safey El Din and Schost to study such projection functions for non-compact connected components can be found in [29].

Related symbolic approaches for computational real algebraic geometry include Sturm's ancient method for counting real roots of a polynomial and Tarski's real quantifier elimination [35, 33] and the work about the number of connected components of a semi-algebraic set [17]. They also include cylindrical algebraic decomposition (CAD) introduced by Collins [13] and improved by Hong [19]. Recent improvement of CAD by using triangular decompositions are given in [12, 11] for solving semi-algebraic systems. But the double exponential cost of the CAD algorithm [14] is the main barrier to its application. For more references see [3], and the references in [29, 30, 5].

In contrast, remarkable developments concerning the computation of real radical of zero dimensional polynomial systems due to Laserre et al [23] are based on moment matrix and numerical semi-definite programming. Recently such moment matrix completion techniques are explored by Zhi et al in [26] for finding at least one real root of a given semi-algebraic system. Furthermore, based on critical point techniques and moment matrix completion, they studied the computation of verified real solutions on components of positive dimensional systems in [38].

The methods of numerical algebraic geometry [34] compute approximate complex points on all irreducible solution components of multivariate complex systems of polynomial equations. Such points on components of each

possible dimension are obtained by slicing with random planes of equal co-dimension. The points are called *witness points* and are computed with efficient homotopy methods. Finding real solution components is a natural extension with many applications. Naive extension of the complex approach by random slicing to the real case fails, since such random planes may not intersect some (e.g. compact) components.

Consider k polynomials from $\mathbb{R}[x_1, ..., x_n]$

$$f = \{f_1, f_2, ..., f_k\} = 0 \tag{1}$$

satisfying the following regularity assumptions:

A_1: $V_{\mathbb{R}}(f_1, f_2, ..., f_i)$ has dimension $n - i$ for $1 \leq i \leq k$.

A_2: the ideal $I_i = \langle f_1, f_2, ..., f_i \rangle$ is radical for $1 \leq i \leq k$.

The solution set or variety of $f = 0$ is:

$$V_{\mathbb{R}}(f_1, ..., f_k) = \{x \in \mathbb{R}^n : f_j(x) = 0,\ 1 \leq j \leq k\} \tag{2}$$

In [37], we construct the following square system to find real witness points

$$F = \{f,\ \sum_{i=1}^{k} \lambda_i \nabla f_i - \mathbf{n}\} = 0 \tag{3}$$

and an $n - k - 1$ dimensional system

$$f^{(1)} = \{f, x \cdot \mathbf{n} - 1\} = 0 \tag{4}$$

Here $\mathbf{n}$ is a random vector in $\mathbb{R}^n$ and (3) has $n + k$ equations and $n + k$ unknowns $(x, \lambda) = (x_1, ..., x_n, \lambda_1, ..., \lambda_k)$. For Equation (4), we can construct a square system $F^{(1)}$ and an $n - k - 2$ dimensional system $f^{(2)}$ by the same idea. Thus, altogether there are $n - k + 1$ square systems to be solved. The last one $F^{(n-k-1)} := f^{(n-k)}$ is of the form $\{f, L_1, ..., L_{n-k}\}$ where each L_i is a linear equation in x with random real coefficients. Therefore the real witness set of f can be expressed as:

$$W_{\mathbb{R}}(f) = V_{\mathbb{R}}(F) \cup W_{\mathbb{R}}(f^{(1)}) = V_{\mathbb{R}}(F) \cup \cdots \cup V_{\mathbb{R}}(F^{(n-k)}) \tag{5}$$

Related critical point techniques include point-distance formulations to obtain real points as the critical points of the distance from a component to a random point [27, 18].

Previously we introduced a perturbation method with the goal of obtaining a real witness point with high accuracy on each real connected solution component of a system. As explained in our earlier work, singularities inevitably occur and it is important to develop approaches to deal with them. The key in our approach to finding well-conditioned witness points is to track a path of a perturbed system to escape the numerically difficult region near the singularity and then project back to the original real variety.

As a sequel of our previous work [37] we will first explore the complexity of computing real points by using the plane-distance construction. Then we will consider numerical aspects of the methods in this paper with the assumption that a real witness point on a positive dimensional component has already been obtained. The main contributions of this paper are two strategies to ensure the success of this method which were not provided in [37]. One is to use adapted step size to avoid jumping to another component during tracking the path by the prediction-projection technique. Another is to determine the direction by which we can escape the singularity as soon as possible.

Shub and Smale's condition number analysis of nonlinear systems can be used to estimate the size of the convergent region of a root and the minimal distance to other roots [15, 9]. Based on it Beltran and Leykin gave a rigorous algorithm for homotopy path-following and its complexity analysis to guarantee convergence and avoid path-jumping [8]. For other important related work on bounds see [16] for root isolation of (sparse) multivariate polynomial systems and see [21] for significant work regarding the minimum distance between two real components.

In this paper, we will introduce an alternative way to estimate the root distance which is one of the key points in our step size control. We assume the convergence of Newton iteration during the projection stage and under this assumption we give a more practical method for avoiding the numerical difficulties. It differs from the analysis of Beltran and Leykin in that a rigorous bound is not given on the step size to ensure convergence while Beltran and Leykin give such a bound for certainty at a higher cost. But our method is guaranteed to detect "path jumping". If jumping or divergence happens, we can decrease the step size and eventually it will work for sufficiently small step size.

2 Complexity Estimation

Tracking a homotopy path is essentially solving an index-1 Differential Algebraic Equation (DAE) with a polynomial cost [20]. Thus, to estimate the total cost of solving system (3) is reduced to counting the number of homotopy paths.

To simplify the root counting we assume that each input polynomial f_i is dense, that is it contains all the terms of degree less than or equal to $d_i = deg(f_i)$. Let $d = \max\{d_1, ..., d_k\}$.

The Bézout bound for Equation (3) is

$$d^n \cdot \prod_{i=1}^{k} d_i \tag{6}$$

which is much greater than the Bézout bound $\prod_{i=1}^{k} d_i$ for the original system.

In fact this root count is often a gross overestimate. To explore a sharp root bound we consider a simple example: a single degree 3 polynomial f in 6 variables. If we apply the construction (3), it leads to a square system $\{f = 0, \lambda f_{x_1} = c_1, ..., \lambda f_{x_6} = c_6\}$, where $c_1, ..., c_6$ are random constants. Its Bézout number is $3^6 \cdot 3 = 2187$. We can eliminate the variable λ first and obtain the system

$$\{f = 0, c_i f_{x_1} = c_1 f_{x_i},\ \lambda f_{x_1} = c_1, \} \text{ for } i = 2, ..., 6.$$

Then we solve for $x_1, ..., x_6$ using the first 6 equations. The corresponding Bézout number is $3 \cdot 2^5$. Finally we substitute x into the last equation, which is linear with respect to λ, to obtain the value of λ. Thus, the number of roots is only 96.

Clearly, the extra cost of solving the system $f^{(1)} = \{f, \sum_{i=1}^{6} c_i\ x_i - 1\} = 0$ should be considered. Eliminating the variable x_6 by using the linear equation yields a single degree 3 polynomial with 5 variables. And repeating the plane-distance construction the total number of roots for all the systems is $3 \cdot (2^5 + 2^4 + \cdots + 2^0) = 189$.

In general, we have the following theorem.

Theorem 2.1 *For a polynomial system $f = \{f_1, ..., f_k\}$ with n variables and degrees $d_i = deg(f_i)$, the number of complex roots of the corresponding*

optimization system (3) is bounded by

$$\binom{n-1}{n-k}(d-1)^{n-k}\prod_{i=1}^{k} d_i \tag{7}$$

where $d = \max\{d_1, ..., d_k\} > 1$ *and* $n > k > 0$.

Moreover, the total number of roots to obtain $W_{\mathbb{R}}(f)$ *in (5) is bounded by*

$$\sum_{j=0}^{j=n-k} \binom{n-1-j}{n-k-j}(d-1)^{n-k-j} \cdot \prod_{i=1}^{k} d_i. \tag{8}$$

Proof.

Consider the square system $\{f, \partial_{x_1}(f \cdot \lambda) - c_1, ..., \partial_{x_n}(f \cdot \lambda) - c_n\}$. It leads to

$$\{f, c_i\partial_{x_1}(f \cdot \lambda) - c_1\partial_{x_i}(f \cdot \lambda)\} \text{ for } i = 2, ..., n.$$

Since the normal vector is chosen at random, λ_1 is nonzero with probability one. Let $v_j = \frac{\lambda_j}{\lambda_1}$ for $j = 2, ..., k$. Changing the variables yields

$$\{f, c_i\partial_{x_1}(f \cdot (1, v)^t) - c_1\partial_{x_i}(f \cdot (1, v)^t)\} \text{ for } i = 2, ..., n. \tag{9}$$

Now we exploit the multi-homogeneous structure associated with the partition x and v. By using the notation of Chapter 8 in [34], the i-th polynomial belongs to $\langle\{1, x\}^{(d_i)} \otimes \{1, v\}^{(0)}\rangle$ when $1 \leq i \leq k$, and the last $n-1$ polynomials belong to $\langle\{1, x\}^{(d)} \otimes \{1, v\}^{(1)}\rangle$. By the multi-homogeneous Bézout Theorem, the total number of roots of (3) is bounded by $\binom{n-1}{n-k}(d-1)^{n-k}\prod_{i=1}^{k} d_i$.

For $W_{\mathbb{R}}(f)$, we first consider $f^{(1)} = \{f, x \cdot \mathbf{n} - 1\}$. This linear equation can be used to eliminate one of the variables and it yields k polynomials of $n-1$ variables with the same degree. By the previous result, the corresponding optimization system $F^{(1)}$ has at most $\binom{n-2}{n-k-1}(d-1)^{n-k-1}\prod_{i=1}^{k} d_i$ roots. The same idea can be applied to $f^{(j)}$. Adding all these expressions together yields Equation (8). □

Actually, the optimization problem (3) is well known and we refer to the work in [32] and [28] for a complete treatment on critical point counting of such so-called "Lagrange polynomial systems".

Our experimental results show this bound for dense systems is optimal. However, when f is sparse, we can use polyhedral homotopy to find the

witness points of f in $\mathbb{C}$ by tracking M homotopy paths, where M is the mixed volume of f with $n-k$ linear equations. Suppose the solution set is denoted by S. Starting from S we can simply use the Cheater's homotopy [24] to produce the solutions of f with $n-k$ linear factors h_{ij} of $\{h_2, ..., h_n\}$ and also the corresponding values of v. Then all the complex solutions of (3) can be obtained by a homotopy connecting the system $\{f, h_2\ell_2, ..., h_n\ell_n\}$ and the target system (3). Thus, a better estimation of the total number of roots is $\binom{n-1}{n-k}(d-1)^{n-k}M$.

Corollary 2.2 *Let $f = \{f_1, ..., f_k\}$ be a polynomial system with n variables. Suppose the complex variety $V(f)$ is a pure $n-k$ dimensional algebraic set with degree M and the degree of each polynomial is bounded by d. Then the total number of roots of Equation (3) is less than or equal to $\binom{n-1}{n-k}(d-1)^{n-k}M$.*

For instance, when $k = n-1$, the variety of f is a curve of dimension 1. To find real witness points (to solve the optimization problem), the total number of homotopy paths is bounded by $(n-1)(d-1)\ M$.

Remark 2.3 *Interestingly, experimental results for a large number of dense systems show that the bound given in Theorem 2.1 is exactly the mixed volume computed by* `Hom4Ps2` *[22].*

However, if f is sparse, the partial derivatives of f are also sparse. Mixed volume techniques should be applied to the whole system (3) to obtain better complexity. For example, consider $f = \{-72\,x_1 - 97\,{x_1}^3 + 33\,x_1\,x_2\,x_3 + 1, -89\,x_1\,x_3 + 65\,{x_1}^3 + 12\,x_1\,{x_2}^2 + 2\}$ with $d = 3, n = 3, k = 2$. The root bound of the corresponding optimization problem given by Theorem 2.1 is 36, whereas the mixed volume of f with a random linear equation is just 6 which lowers the bound down to $(3-1) \times (3-1) \times 6 = 24$. A more sophisticated mixed volume computation applied to the whole optimization problem by `Hom4Ps2` *shows that there are only 15 roots.*

3 Error Control

For numerical stability, we homogenize the system f by adding a variable x_0 satisfying a new equation $\sum_{i=0}^{n} x_i^2 = 1$. The homogenized system $\bar{f}$ is homogenous except for the new inhomogeneous equation.

Now we estimate the maximum value of a given polynomial on the hypersphere $\sum x_i^2 = 1$. Let $\alpha = (a_0, ..., a_n)$ be the exponent of a monomial, then we define

$$\alpha^\alpha = a_0^{a_0} \cdots a_n^{a_n} \text{ and } 0^0 = 1 \, .$$

Proposition 3.1 *Given a homogenous polynomial $g = \sum_{i=1}^{T} c_i \boldsymbol{x}^{\alpha_i}$ of degree d, where $\boldsymbol{x} = (x_0, ..., x_n)$, an upper bound of $|g|$ on $\sum_{i=0}^{n} x_i^2 = 1$ in $\mathbb{R}^{n+1}$ is $\sum_{i=1}^{T} \mid c_i \mid \sqrt{\frac{\alpha_i^{\alpha_i}}{d^d}}$ denoted by $S(g)$.*

Proof. Without loss of generality we assume each $\alpha_i > 0$. To attain the maximum value of a monomial $\boldsymbol{x}^\alpha = x_0^{a_0} \cdots x_n^{a_n}$ on $\sum x_i^2 = 1$, we use the method of Lagrange multipliers $\nabla \boldsymbol{x}^\alpha = \lambda \nabla \sum x_i^2$, i.e.

$$(a_0 \boldsymbol{x}^\alpha / x_0, ..., a_n \boldsymbol{x}^\alpha / x_n) = 2\lambda (x_0, ..., x_n)$$

This implies that $\frac{x_0^2}{a_0} = \frac{x_1^2}{a_1} = \cdots = \frac{x_n^2}{a_n}$. Since $\sum x_i^2 = 1$, we have

$$x_i^2 = \frac{a_i}{\sum_{j=0}^{n} a_j} = \frac{a_i}{d} \, .$$

So the maximum value of the monomial $\boldsymbol{x}^\alpha$ on $\sum x_i^2 = 1$ is

$$\sqrt{\frac{\prod_0^n a_i^{a_i}}{d^d}} = \sqrt{\frac{\alpha^\alpha}{d^d}}$$

Thus, $|g| = |\sum c_i \boldsymbol{x}^{\alpha_i}| \leq \sum |c_i| \, |\boldsymbol{x}^{\alpha_i}| \leq \sum \mid c_i \mid \sqrt{\frac{\alpha_i^{\alpha_i}}{d^d}}$. □

Remark 3.2 *For a polynomial system $g = \{\bar{f}_1, ..., \bar{f}_k, \sum_{t=0}^{n} x_t^2/2 - 1/2\}$, where $\bar{f}_i$ is homogenous for $1 \leq i \leq k$. Let $J_{ij} = \partial \bar{f}_i / \partial x_j$ which is a polynomial for $1 \leq i \leq k$, and $J_{ij} = x_j$ for $i = k+1$. So the gradient $v_{ij} = \nabla J_{ij}$ is a vector of polynomials and $v_{ij} \cdot v_{ij} = \|\nabla J_{ij}\|_2^2$ is a homogenous polynomial. Applying Proposition 3.1 yields an upper bound $\sqrt{S(v_{ij} \cdot v_{ij})}$ for $\|\nabla J_{ij}\|_2$. Define*

$$K(g) = \max\{\sqrt{S(v_{ij} \cdot v_{ij})} : 1 \leq i \leq k+1 \text{ and } 0 \leq j \leq n\}. \qquad (10)$$

We simply denote it by K if there is no ambiguity.

Example 3.1 *Consider the implicit curve* $f = \{y^2 + z^2 - (2\ x - x^2)^3 = 0, z - y^2 = 0\}$ *given in [37]. The corresponding system* g *is* $\{u^4y^2 + u^4z^2 - 8\,u^3x^3 + 12\,u^2x^4 - 6\,ux^5 + x^6, uz - y^2, (x^2 + y^2 + z^2 + u^2 - 1)/2\}$. *So the Jacobian is*

$$\begin{bmatrix} J_{11} & 2\,u^4y & J_{13} & 2\,u^4z \\ 0 & -2\,y & z & u \\ x & y & u & z \end{bmatrix}$$

where $J_{11} = -24\,u^3x^2 + 48\,u^2x^3 - 30\,ux^4 + 6\,x^5$ *and* $J_{13} = 4\,u^3y^2 + 4\,u^3z^2 - 24\,u^2x^3 + 24\,ux^4 - 6\,x^5$.

Using Proposition 3.1 to find the upper bounds of the 2-norm of the gradient of each $\mathcal{J}_{ij}$ *yields*

$$\begin{bmatrix} 144.2345348 & 10.44030651 & 92.45156154 & 0.5724334022 \\ 0 & 2.0 & 1.0 & 1.0 \\ 1.0 & 1.0 & 1.0 & 1.0 \end{bmatrix}$$

Thus $K(g) = 144.2345348$. □

3.1 Root Isolation Balls

For a square system $g(x) = \{f_1, ..., f_n, \sum_{i=0}^n x_i^2/2 - 1/2\}$ with $n+1$ variables, each f_i is homogenous. Suppose its maximum value of $\|\nabla J_{ij}\|_2$ for all i, j on $\sum_{i=0}^n x_i^2 = 1$ is bounded by $K = K(g)$ which can be obtained by Proposition 3.1.

First we assume there are at least two real roots $\boldsymbol{z}_1, \boldsymbol{z}_2$ and we will estimate the distance between $\boldsymbol{z}_1, \boldsymbol{z}_2$. For an estimate of global minimal distance for all the roots, we refer to the significant work by Emiris et al [16]. But here we only need local information for a specific root and hopefully it will be more practical in numerical computation.

Let $\boldsymbol{z}_2 = \boldsymbol{z}_1 + \boldsymbol{h}$ and σ'_{n+1} be the smallest singular value of the Jacobian matrix of g at $\boldsymbol{z}_1$.

By the multivariate fundamental theorem of calculus,

$$g(\boldsymbol{z}_1 + \boldsymbol{h}) - g(\boldsymbol{z}_1) = \left(\int_0^1 \mathcal{J}(\boldsymbol{z}_1 + t\boldsymbol{h}) dt \right) \cdot \boldsymbol{h} = 0$$

where $\mathcal{J}$ is the Jacobian matrix of $g(z)$. So the matrix $D = \int_0^1 \mathcal{J}(\boldsymbol{z}_1 + t\boldsymbol{h})dt$ must be singular and $\|\mathcal{J}(\boldsymbol{z}_1) - D\|_2 \geq \sigma'_{n+1}$ by Eckart-Young Theorem. And consequently

$$\|\mathcal{J}(\boldsymbol{z}_1) - D\|_{\max} \geq \|\mathcal{J}(\boldsymbol{z}_1) - D\|_2/(n+1) \geq \sigma'_{n+1}/(n+1). \tag{11}$$

We remark that if there are only $k+1$ nonlinear equations in g then there are only $k+1$ nonlinear rows in the matrix $\mathcal{J}(\boldsymbol{z}_1) - D$. Thus, the inequality (11) becomes $\|\mathcal{J}(\boldsymbol{z}_1) - D\|_{\max} \geq \|\mathcal{J}(\boldsymbol{z}_1) - D\|_2/\mu \geq \sigma'_{n+1}/\mu$, where $\mu = \sqrt{(k+1)(n+1)}$.

Applying the multivariate mean value theorem to $\mathcal{J}_{ij}(\boldsymbol{z}(t))$ yields

$$\int_0^1 \mathcal{J}_{ij}(\boldsymbol{z}_1 + t\boldsymbol{h})dt = \mathcal{J}_{ij}(\boldsymbol{z}_1) + \nabla\mathcal{J}_{ij}(c) \cdot \boldsymbol{h},$$

where $c = \boldsymbol{z}_1 + \zeta\boldsymbol{h}$ for some $\zeta \in (0,1)$. Thus,

$$\sigma'_{n+1}/(n+1) \leq \|\mathcal{J}(\boldsymbol{z}_1) - D\|_{\max} \leq \max_{ij} \|\nabla\mathcal{J}_{ij}\|_2 \|\boldsymbol{h}\|_2 \leq K\|\boldsymbol{h}\|_2$$

and consequently

$$\|\boldsymbol{h}\|_2 \geq \frac{\sigma'_{n+1}}{(n+1)K} \tag{12}$$

The estimated distance between two roots is $\frac{\sigma'_{n+1}}{(n+1)K}$. In other words, there is only one real root isolated in the ball centered at $\boldsymbol{z}_1$ with radius $\frac{\sigma'_{n+1}}{(n+1)K}$. This ball is called a *root isolating ball* of $\boldsymbol{z}_1$, and its radius is called an *isolating radius* for $\boldsymbol{z}_1$.

To summarize the result above, we have the following lemma.

Lemma 3.3 *[Root Isolation Lemma] Let σ'_{n+1} be the smallest singular value of the Jacobian of g at $\boldsymbol{z}_1$. Then $\frac{\sigma'_{n+1}}{(n+1)K(g)}$ is an isolating radius for $\boldsymbol{z}_1$, where $K(g)$ is defined in Equation(10). Moreover, if there are only $k+1$ nonlinear equations in g, then an isolating radius is $\frac{\sigma'_{n+1}}{\mu K(g)}$, where $\mu = \sqrt{(k+1)(n+1)}$.*

Obviously the isolating radius is often underestimated. For a better lower bound on isolating radii, we consider a rescaling of the system g. The upper bound matrix for $\nabla\mathcal{J}_{ij}$ is denoted by $\mathfrak{G} = \begin{pmatrix} \mathfrak{g}_1 \\ \vdots \\ \mathfrak{g}_{n+1} \end{pmatrix}$ where $\mathfrak{g}_i$ is a row vector.

Let $\theta_i = \|\mathfrak{g}_i\|_\infty$. Then $K(g) = \max\{\theta_1, ..., \theta_{n+1}\}$. Define

$$\tilde{g} = \begin{pmatrix} 1/\theta_1 & & \\ & \ddots & \\ & & 1/\theta_{n+1} \end{pmatrix} \cdot g$$

Then the new system is called a *normalized system* for g and $K(\tilde{g}) = 1$.

Interestingly we will show that $\tilde{g}$ has a larger isolating radius estimate.

Proposition 3.4 *Let σ'_{n+1} and $\tilde{\sigma}_{n+1}$ be the smallest singular value at $\boldsymbol{z}_1$ of the Jacobian of g and its normalized system $\tilde{g}$ respectively. Then*

$$\frac{\sigma'_{n+1}}{K(g)} \leq \frac{\tilde{\sigma}_{n+1}}{K(\tilde{g})}$$

PROOF. Consider $K\tilde{g} = \begin{pmatrix} K/\theta_1 & & \\ & \ddots & \\ & & K/\theta_{n+1} \end{pmatrix} \cdot g$, where $K = K(g)$. The elements of the diagonal matrix Λ are greater than or equal to one.

Let $\mathcal{J}$ be the Jacobian of g at $\boldsymbol{z}_1$. Then $\Lambda \cdot \mathcal{J}$ is the Jacobian of $K\tilde{g}$ at $\boldsymbol{z}_1$. By the definition, there exists a unit vector x_0 such that

$$\sigma'_{n+1} = \min_{\|x\|=1} \|\mathcal{J}x\|_2 = \|\mathcal{J}x_0\|_2$$

Similarly, there exists a unit vector y_0 such that

$$K\tilde{\sigma}_{n+1} = \min_{\|y\|=1} \|\Lambda\mathcal{J}y\|_2 = \|\Lambda\mathcal{J}y_0\|_2$$

Thus, $K\tilde{\sigma}_{n+1} = \|\Lambda\mathcal{J}y_0\|_2 \geq \|\mathcal{J}y_0\|_2 \geq \|\mathcal{J}x_0\|_2 = \sigma'_{n+1}$. □

Example 3.2 *Reconsider the previous example $g = \{u^4y^2 + u^4z^2 - 8\,u^3x^3 + 12\,u^2x^4 - 6\,ux^5 + x^6, uz - y^2, x^2 + y^2 + z^2 + u^2 - 1, z - 0.01\}$ with $K = 144.2$. It is easy to show $F = f(x, y, 0.01, u)$ has 4 real roots when $z = 0.01$. We want to estimate the distance of the root $p = (u = 0.988, y = -0.0994, x = 0.114)$ to other roots.*

The smallest singular value of the Jacobian at p is 0.141. Thus, we obtain an isolating radius of 0.000345.

On the other hand, since $\{\theta_1 = 144.23, \theta_2 = 2, \theta_3 = 2, \theta_4 = 1\}$, we construct $\tilde{g} = \{g_i/\theta_i\}$ and the resulting isolating radius is 0.000546 for the normalized system.

The condition number type of isolating radius was given by Blum, Cucker, Shub and Smale [9] and it was improved by Jean-Pierre Dedieu in [15]. To illustrate the difference, we consider the following example.

Example 3.3 *Let f be a linear homogenous system with $n+1$ variables and n equations. Suppose all the coefficients are unit complex numbers, the 2-norm of the solution x is also one and the smallest singular value of the Jacobian is $\sigma_n < 1$. Then the norm of the homogenous system f defined in [9] is $\sqrt{n(n+1)}$ and consequently an isolating radius is given by $sep_{proj}(f,x) \geq \sigma_n/\sqrt{n(n+1)}$ in the sense of "angle distance" $d_p(x,y)$ in Equation (2.4) of [15].*

To use our result, let $g = \{f, (\sum x_i^2 - 1)/2\}$. So $K(g) = 1$. By Proposition 3.8 the smallest singular value of the Jacobian of g is still σ_n. And by Lemma 3.3 an isolating radius of x is $\sigma_n/\sqrt{n+1}$ in the sense of the standard 2-norm since there is only one nonlinear equation.

Note that for any two roots x and y on the sphere $\sum x_i^2 - 1 = 0$ we have $\|x-y\|_2 = 2\sin\frac{\theta}{2} \geq \sin\theta = d_p(x,y)$. These two norms are very close if the angle θ is small, i.e. x and y are close to each other. For this case, our method can find a better root isolation ball.

Remark 3.5 *Our root isolation method only needs local information at this point. Proposition 3.1 gives a more refined "measure" of a polynomial than traditional polynomial "height" or "norm". The upper bound K only depends on the system itself, so we only need to compute K once. For the system $g = \{\bar{f}, (\sum x_i^2 - 1)/2\}$, its K is greater than or equal to 1. Thus, even if the θ_i corresponding to g_i is less than 1, the result is still valid even if we do not rescale g_i. This fact will be used in the later sections.*

The estimate of K is not optimal here. More sophisticated analysis is needed for improvement of step size control. In fact, the condition number type of isolating radius can be also applied to the rest of the paper to obtain a similar result which warrants further study.

3.2 Direction Determination

We aim to find a "good" path from an ill-conditioned point on a positive dimensional component leading to a well-conditioned region.

Previously, we have shown that root isolating balls have a close relation with the smallest singular value. Naturally, the key question to be answered

in this section is to find good directions in higher (> 1) dimensional tangent spaces in order to increase the smallest singular values along the path on the component. Such a path should lead to a well-conditioned point on the perturbed component which allows us to find a point on the original system with higher accuracy by projection.

Let $f = \{f_1, ..., f_k\}$ be a polynomial system with n variables. Let $\mathcal{J}$ be the Jacobian of a system f at $\boldsymbol{z}_0$ with $\mathcal{J} = U\Sigma V^t$. Let $A = \mathcal{J}\mathcal{J}^t = U\Sigma^2 U^t$. Thus, $A \cdot u = \sigma_k^2 u$ and $u^t u = 1$ where σ_k and u are the smallest singular value and the corresponding singular vector in U. For brevity, we write σ for σ_k^2 in this subsection.

Imposing a small perturbation on the equations above yields

$$(A + \Delta A)(u + \Delta u) = (\sigma + \Delta\sigma)(u + \Delta u) \text{ and } (u + \Delta u)^t(u + \Delta u) = 1 \quad (13)$$

Ignoring the higher order terms, we have

$$\begin{aligned} Au + \Delta A u + A\Delta u &= \sigma u + \Delta\sigma u + \sigma\Delta u \\ u^t\, \Delta u &= 0 \end{aligned}$$

and

$$\Delta A u = \big((\mathcal{J} + \Delta\mathcal{J})(\mathcal{J}^t + \Delta\mathcal{J}^t) - A\big)\, u = (\Delta\mathcal{J}\mathcal{J}^t + \mathcal{J}\Delta\mathcal{J}^t)u \qquad (14)$$

where $\Delta\mathcal{J} = \sum_i \frac{\partial \mathcal{J}}{\partial x_i}\Delta x_i$. Thus, we obtain an $(m + 1 + m) \times (1 + m + n)$ homogenous linear system:

$$\begin{pmatrix} u & (\sigma I - A) & -A_1 & \cdots & -A_n \\ & u^t & & & \\ & & & \mathcal{J} & \end{pmatrix} \cdot \begin{pmatrix} \Delta\sigma \\ \Delta u \\ \Delta x_1 \\ \vdots \\ \Delta x_n \end{pmatrix} = 0 \qquad (15)$$

where $A_i = (\frac{\partial \mathcal{J}}{\partial x_i}\mathcal{J}^t + \mathcal{J}\frac{\partial \mathcal{J}}{\partial x_i}^t)u$.

Lemma 3.6 *If the smallest singular value σ_k of $\mathcal{J}$ is strictly less than other singular values. Then the inverse of* $\begin{pmatrix} u & (\sigma I - A) \\ 0 & u^t \end{pmatrix}$ *is* $\begin{pmatrix} u^t & 0 \\ B & u \end{pmatrix}$, *where*

$$B = U \begin{pmatrix} 1/(\sigma - \sigma_1^2) & & & \\ & \ddots & & \\ & & 1/(\sigma - \sigma_{m-1}^2) & \\ & & & 0 \end{pmatrix} U^t$$

where $\sigma = \sigma_k^2$.

PROOF. It is straightforward to verify that $Bu = 0$ and $u^t(\sigma I - A) = 0$. Now we will show that $B(\sigma I - A) + uu^t = I$.

Consider $(B(\sigma I - A) + uu^t)UU^t$ which is equal to $\sigma BUU^t - BAUU^t + uu^tUU^t$.

By substituting the SVDs of A and B, we have

$$U\Sigma_1 U^t - U\Sigma_2 U^t + U\Sigma_3 U^t = I$$

where $\Sigma_1 = \text{diag}\left(\frac{\sigma}{\sigma-\sigma_1^2}, \cdots, \frac{\sigma}{\sigma-\sigma_{m-1}^2}, 0\right)$,
$\Sigma_2 = \text{diag}\left(\frac{\sigma_1^2}{\sigma-\sigma_1^2}, \cdots, \frac{\sigma_{m-1}^2}{\sigma-\sigma_{m-1}^2}, 0\right)$ and $\Sigma_3 = \text{diag}(0, \cdots, 0, 1)$. □

Equation (15) multiplied by $\begin{pmatrix} u^t & & \\ B & u & \\ & & I \end{pmatrix}$ gives

$$\Delta\sigma = u^t A_1 \Delta x_1 + \cdots + u^t A_n \Delta x_n. \tag{16}$$

Suppose the step size is δ. A good direction aiming to maximize $\Delta\sigma$ leads to the following optimization problem:

$$\begin{array}{ll} \max & c \cdot \Delta x \\ \text{s.t.} & \mathcal{J} \cdot \Delta x = 0 \\ & \|\Delta x\|_2 = \delta \end{array} \tag{17}$$

where $c = (u^t A_1, ..., u^t A_n)$ and $\Delta x = (\Delta x_1, ..., \Delta x_n)^t$.

Lemma 3.7 *Let u be the singular vector corresponding to the smallest singular value, and $A_i = (\frac{\partial \mathcal{J}}{\partial x_i}\mathcal{J}^t + \mathcal{J}\frac{\partial \mathcal{J}^t}{\partial x_i})u$ and $c = (u^t A_1, ..., u^t A_n)$. If $c \notin \langle \mathcal{J} \rangle$, then the solution to the optimization problem (17) is*

$$\Delta x = \frac{Hc^t}{\sqrt{cHc^t}}\,\delta \quad \textit{and} \quad \Delta\sigma = \sqrt{cHc^t}\,\delta \tag{18}$$

where $H = I - \mathcal{J}^t(\mathcal{J}^t)^\dagger$, and $(\mathcal{J}^t)^\dagger$ is the Moore-Penrose pseudo-inverse of $\mathcal{J}^t$.

# steps	σ	point p
1	0.069	(-0.00072, 0.031, 0.016)
2	0.071	(0.019,0.032,0.016)
3	0.087	(0.038, 0.036, 0.018)
4	0.12	(0.055,0.044,0.022)
5	0.16	(0.071, 0.055, 0.027)
6	0.21	(0.086,0.067,0.033)
7	0.26	(0.099, 0.08, 0.04)
8	0.32	(0.11,0.093,0.047)
9	0.38	(0.13, 0.11, 0.053)
10	0.44	(0.14,0.12,0.06)

Table 1: Our path-tracking technique with fixed step size 0.02 leads to a well-conditioned new point. The column σ gives singular values at each point p.

PROOF. It is easy to check that $H^t = H$, $H^2 = H$, $H\mathcal{J}^t = 0$ and $\mathcal{J}^\perp H = \mathcal{J}^\perp$, where $\mathcal{J}^\perp$ is the orthogonal complement of $\mathcal{J}$. Thus, $\Delta x \in \langle \mathcal{J}^\perp \rangle$ belongs to the space spanned by H. And $H(c^t - Hc^t) = 0$ implies that Hc^t is the projection of c in $\langle H \rangle$ and $\Delta x = \lambda Hc^t$ will maximize the inner product $c \cdot \Delta x$, where is λ is some positive constant to be determined. Since $\delta = \|\Delta x\|_2 = \lambda\sqrt{cH^tHc^t}$, we have $\lambda = \delta/\sqrt{cHc^t}$ and $\Delta\sigma = c \cdot \Delta x = \sqrt{cHc^t}\ \delta$.

In addition, $Hc^t = 0$ if and only if $c \in \langle \mathcal{J} \rangle$. Therefore, we obtain the result (18). □

We require that the Jacobian at $\boldsymbol{z}_0$ satisfies the "strictly less than" condition in Lemma 3.6 and "$c \notin \langle \mathcal{J} \rangle$" condition in Lemma 3.7. Actually, these conditions hold with high probability because of the randomness in Equation (3). We will pick another random vector $\mathbf{n}$ to produce other critical points if the singular values are close.

Example 3.4 *Again we use the implicit curve* $f = \{y^2 + z^2 - (2\,x - x^2)^3 - 0.0012\}$ *to illustrate our method. We focus on the real variety of* f *near the origin. Suppose we have a starting point* $p = (x = -0.02, y = 0.03, z = 0.015)$. *The singular value at* p *is* 0.068. *Fix the step size to be* 0.02. *By Lemma 3.7, the direction for the next step is the unit vector* $(.988, .138, 0.0689)$ *and the new point is* $(-0.00072, 0.031, 0.016)$. *From this point, we can track to additional point on this component by the same idea.*

We use Table 1 to show the changes of the singular values in the first 10 *steps. It indicates that our method can find a new point* $(0.14, 0.12, 0.06)$ *which has better numerical behavior.*

3.3 Step Size Control

In previous example, we fixed the step size during the path tracking. However, it may lead to a "path jumping" if there is another component nearby. So we discuss an adaptive step size strategy in this section.

For a given system $f(x_1, ..., x_n) = \{f_1, ..., f_k\}$ satisfying regularity assumptions in (1), we homogenize f and suppose its normalized system is

$$g(x_0, x_1, ..., x_n) = \{g_1, ..., g_k, (\sum x_i^2 - 1)/2\}$$

with a point $\boldsymbol{z}_0$ on the hyper-sphere $S(0; 1)$. Suppose the dimension $m = n-k$ is greater than one.

By the result of the previous section, the direction is Hc^t which is in the tangent space. Let S be the orthogonal complement of the vector Hc^t, so S has dimension n. By the regularity assumptions, the space spanned by $\mathcal{J}$ of dimension $k+1$ is contained in S. Let $S = \mathcal{J} \oplus T$ where T consists of $m-1$ orthogonal vectors (represented by an $(m-1) \times (n+1)$ matrix) with unit 1-norm. Thus, at $\boldsymbol{z}_0$, tracking in the tangent direction Hc^t is equivalent to tracking the 1-dimensional system

$$G = \{g, Tx - T\boldsymbol{z}_0\}. \tag{19}$$

It is easy to check that $K = K(G) = 1$ and here we do not rescale the new equations $Tx - T\boldsymbol{z}_0$. Interestingly, the smallest singular value of $\mathcal{J}(G)$ is the same as that of $\mathcal{J}(g)$ at $\boldsymbol{z}_0$.

Proposition 3.8 *Suppose A be an $m \times n$ matrix with smallest singular value $0 < \sigma_m < 1$. Suppose a matrix $B_{t\times n}$ satisfies $AB^t = 0$ and $BB^t = I$ and $m + t \leq n$. Then the smallest singular value of the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ is σ_m.*

PROOF. Let the singular value decomposition of A be $U_{m\times m} \begin{pmatrix} \Sigma_{m\times m} & 0 \end{pmatrix} \begin{pmatrix} V_1^t \\ V_2^t \end{pmatrix}$. Thus, V_2^t is the orthogonal complement of A, and consequently B spans a

subspace of V_2^t. Let $\langle V_2^t \rangle = \langle B \rangle \oplus \langle C \rangle$, where C is an orthogonal basis. Then we have the following SVD

$$\begin{pmatrix} A \\ B \end{pmatrix}_{(m+t)\times n} = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}_{(m+t)\times(m+t)} \begin{pmatrix} \Sigma & 0 & 0 \\ 0 & I & 0 \end{pmatrix}_{(m+t)\times n} \begin{pmatrix} V_1^t \\ B \\ C \end{pmatrix}_{n\times n}$$

So the new singular values are 1 and the smallest one is still σ_m. □

Let the Jacobian of G be $\mathcal{J}$. Applying the mean value theorem to each element of $\mathcal{J}$ on $S(0;1)$ yields

$$\mathcal{J}_{ij}(\boldsymbol{y}) = \mathcal{J}_{ij}(\boldsymbol{x}) + \nabla\mathcal{J}_{ij}(\zeta_{ij}) \cdot (\boldsymbol{y} - \boldsymbol{x}),\ \forall \boldsymbol{x}, \boldsymbol{y} \in S(0;1) \tag{20}$$

where $\zeta_{ij} = \boldsymbol{x} + \kappa_{ij}(\boldsymbol{y} - \boldsymbol{x})$ and $\|\nabla\mathcal{J}_{ij}(\zeta_{ij})\|_2 \leq K = 1$ for $i \leq k+1$, $\|\nabla\mathcal{J}_{ij}(\zeta_{ij})\|_2 = 0$ for $i > k+1$.

Denote $\big(\nabla\mathcal{J}_{ij}(\zeta_{ij}) \cdot (\boldsymbol{y} - \boldsymbol{x})\big)_{n\times(n+1)} = \triangle\mathcal{J}$, then we have

$$\begin{aligned} \|\triangle\mathcal{J}\|_2 &= \|\big(|\nabla\mathcal{J}_{ij}(\zeta_{ij}) \cdot (\boldsymbol{y} - \boldsymbol{x})|\big)_{n\times(n+1)}\|_2 \\ &\leq \|\big(\|\nabla\mathcal{J}_{ij}(\zeta_{ij})\|_2 \|\boldsymbol{y} - \boldsymbol{x}\|_2\big)_{n\times(n+1)}\|_2 \\ &\leq \sqrt{(k+1)(n+1)}\|\boldsymbol{y} - \boldsymbol{x}\|_2 \end{aligned} \tag{21}$$

Since G is one dimensional system, a natural direction is the tangent of the curve at $\boldsymbol{z}_0$. In the *predictor* step, we move distance δ in this direction and in the *corrector* step, the Newton method converges to $\boldsymbol{z}_1$. Next we will estimate the distance between $\boldsymbol{z}_0$ and $\boldsymbol{z}_1$.

Let $\mathcal{J}(\boldsymbol{z}_0) = U\Sigma V^t$. Introduce a new coordinate system $u = (u_0, ..., u_n)^t = V^t x$ with its origin at $\boldsymbol{z}_0$.

For a point $\boldsymbol{z}$ on the curve of $V_{\mathbb{R}}(G)$ close to $\boldsymbol{z}_0$, we have

$$0 = \mathcal{J}(\boldsymbol{z})dx = (\mathcal{J}(\boldsymbol{z}_0) + \triangle\mathcal{J})dx = U\Sigma du + \triangle\mathcal{J}V du \tag{22}$$

where dx is the tangent direction at $\boldsymbol{z}$.

And consequently it implies

$$(I, 0)du + \Sigma^{-1}U^{-1}\triangle\mathcal{J}V du = 0 \tag{23}$$

where $\Sigma^{-1} = \begin{pmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n \end{pmatrix}$ and $\triangle\mathcal{J}$ is a function of u.

Denote $\Sigma^{-1}U^{-1}\triangle\mathcal{J}V = (E_{n\times n}, b_{n\times 1})$. Then the 2-norm of $\|E\|_2$ and $\|b\|$ are less than or equal to $\|u\|\sqrt{(k+1)(n+1)}/\sigma_n$ by Equation (21).

Consider the coordinate u_n as the independent variable to describe the curve locally. Split $du = (du_0, ..., du_{n-1}, du_n) = (d\vec{u}, du_n)$, where $\vec{u}$ is a vector function of u_n. Then Equation (23) becomes

$$(I+E)d\vec{u} = -b\, du_n \Rightarrow \|d\vec{u}\| \leq \frac{\|u\|\sqrt{(k+1)(n+1)}}{\sigma_n - \|u\|\sqrt{(k+1)(n+1)}}\,|du_n| \qquad (24)$$

due to the fact $\|(I+E)^{-1}\|_2 \leq 1/(1-\|E\|_2)$ when $\|E\|_2 < 1$.

Denote u_n and $\|\vec{u}\|$ by x and $y(x)$ respectively. Let $\mu = \sqrt{(k+1)(n+1)}$. Equation (24) becomes

$$y' \leq \frac{\mu\sqrt{x^2+y^2}}{\sigma_n - \mu\sqrt{x^2+y^2}} \text{ and } y(0) = 0 \qquad (25)$$

When x is sufficiently small, $\sigma_n - \mu\sqrt{x^2+y^2} > 0$ and $x > 0, y > 0$, then $\sqrt{x^2+y^2} < x + y$ and

$$0 \leq y' < \frac{\mu(x+y)}{\sigma_n - \mu(x+y)} \text{ and } y(0) = 0\,. \qquad (26)$$

It is easy to show that the solution to the IVP $z' = \frac{\mu(x+z)}{\sigma_n - \mu(x+z)}$ and $z(0) = 0$ is

$$z = \frac{\sigma_n}{\mu} - x - \frac{\sqrt{\sigma_n^2 - 2\sigma_n\mu\, x}}{\mu} \text{ when } x \leq \frac{\sigma_n}{2\mu} \qquad (27)$$

Thus, $y(x)$ is bounded above by $z(x)$ when $x \leq \frac{\sigma_n}{2\mu}$.

We set the step size $\delta = \frac{\sigma_n}{2\rho\mu}$, $\rho \geq 1$ and substitute $x = \delta$ in Equation (27). Then

$$\|\boldsymbol{z}_1 - \boldsymbol{z}_0\| < \sqrt{z^2+x^2} = \sqrt{2\,(2\,\rho-1)\left(2\,\rho - 2\,\sqrt{\rho\,(\rho-1)} - 1\right)}\cdot\delta \qquad (28)$$

Denote $\sqrt{2\,(2\,\rho-1)\left(2\,\rho - 2\,\sqrt{\rho\,(\rho-1)} - 1\right)}$ by ω.

Now we discuss how to choose the step size and how to determine if a jump happens.

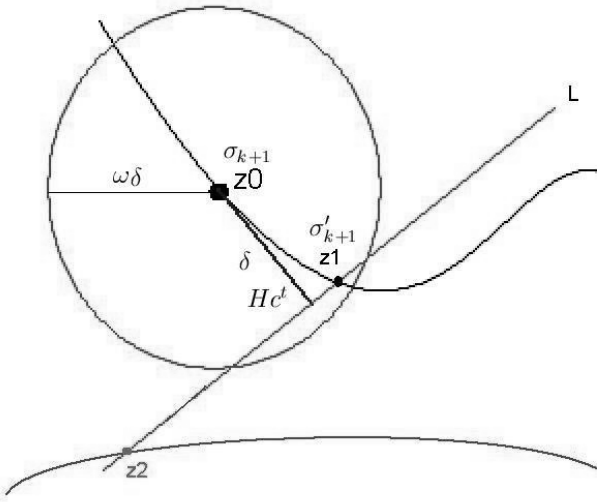

Figure 1: The point z_0 moves δ in the direction Hc^t and $\mathit{dist}\,(\,\boldsymbol{z}_1, \boldsymbol{z}_0\,) < \omega\delta$

By using the same notation, we track the 1-dimensional system $G = \{g, Tx - T\boldsymbol{z}_0\}$ at $\boldsymbol{z}_0$. In order to use the Root Isolation Lemma, we can reduce the system to a zero dimensional one by cutting the curve of G with a hyper-plane which is perpendicular to the tangent Hc^t at distance δ to $\boldsymbol{z}_0$. The equation for this hyper-plane is

$$L = (cH(x - \boldsymbol{z}_0) - cHc^t\delta)/\sqrt{cHc^t}. \tag{29}$$

At each step we monitor the smallest singular value of the Jacobian of g. Note that we do not construct G since the smallest singular values of $\mathcal{J}(g)$ and $\mathcal{J}(G)$ are same by Proposition 3.8.

Theorem 3.9 *Let $g = \{g_1, ..., g_k, (\sum x_i^2 - 1)/2\}$ be a normalized system. Let $\delta = \frac{\sigma_{k+1}}{2\rho\mu}$ where σ_{k+1} is the smallest singular value of the Jacobian of g at $\boldsymbol{z}_0$ and ρ satisfies that $2\rho > 3\omega$ (see Remark 3.10). We move $\boldsymbol{z}_0$ in the direction Hc^t in distance δ and apply the Newton Iteration to the zero dimensional system $\{G, L\}$. Assume the Newton Iteration converges to $\boldsymbol{z}_1$. Then $\boldsymbol{z}_1$ is on the same component with $\boldsymbol{z}_0$ if and only if $\mathit{dist}\,(\,\boldsymbol{z}_1, \boldsymbol{z}_0\,) < \omega\delta$.*

Proof. "⇒" is simply due to Equation (28).

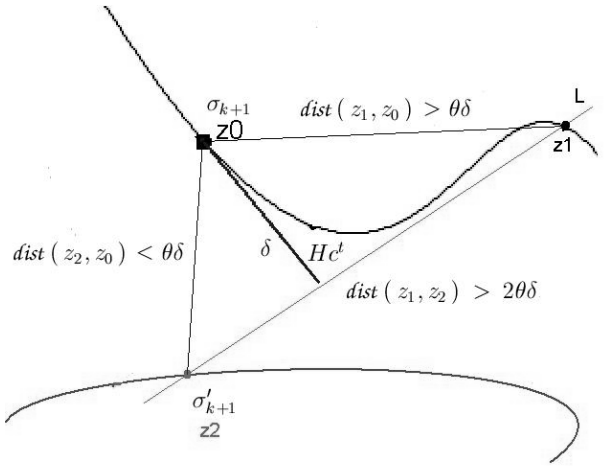

Figure 2: When z_0 moves δ in the direction Hc^t and $dist\,(\,\boldsymbol{z}_2, \boldsymbol{z}_0\,) < \omega\delta$

Now we show "$\Leftarrow$".

Let s_{n+1} and s'_{n+1} be the smallest singular values of the Jacobian of the square system $\{G, L\}$ at $\boldsymbol{z}_0$ and $\boldsymbol{z}_1$ respectively. By Proposition 3.8, $\sigma_{k+1} = s_{n+1}$.

We estimate the s'_{n+1} by Weyl's theorem [36],

$$|s'_{n+1} - \sigma_{k+1}| = |s'_{n+1} - s_{n+1}| \leq \|\triangle\mathcal{J}\|_2 < \mu\omega\delta = \frac{\omega}{2\rho}\sigma_{k+1}$$

Thus, $s'_{n+1} > (1 - \frac{\omega}{2\rho})\sigma_{k+1}$. By the Root Isolation Lemma 3.3, we have

$$dist\,(\,\boldsymbol{z}_1, \boldsymbol{z}_2\,) > s'_{n+1}/\mu > (1 - \frac{\omega}{2\rho})\sigma_{k+1}/\mu$$

Since $2\rho > 3\omega \Leftrightarrow (1 - \frac{\omega}{2\rho})\sigma_{k+1}/\mu > 2\omega\delta$, by our assumption it leads to $dist\,(\,\boldsymbol{z}_1, \boldsymbol{z}_2\,) > 2\omega\delta$.

Combined with the condition $dist\,(\,\boldsymbol{z}_1, \boldsymbol{z}_0\,) < \omega\delta$, we have

$$dist\,(\,\boldsymbol{z}_2, \boldsymbol{z}_0\,) > \omega\delta$$

It means that the distance of $\boldsymbol{z}_0$ to other roots of $\{G, L\}$, which are the intersections of L and other components, must be greater than $\omega\delta$ as shown in Figure 1. So there is only one root within distance $\omega\delta$. If this root belongs

to other components as shown in Figure 2, then $dist(\boldsymbol{z}_1, \boldsymbol{z}_0) > \omega\delta$ which is a contradiction to Equation (28).

Therefore, the root $\boldsymbol{z}_1$ is on the same component with $\boldsymbol{z}_0$. □

Remark 3.10 *The condition $2\rho > 3\omega$ is a theoretical result, and in practice we may choose ρ as small as possible e.g. $\rho = 1.6$ and then $\omega = 1.0285$. Because we do not know the convergence region of Newton iteration, $dist(\boldsymbol{z}_1, \boldsymbol{z}_0)$ may be greater that the bound $\omega\delta$. In that case, we will decrease the step size. Eventually, $dist(\boldsymbol{z}_1, \boldsymbol{z}_0)$ will be less than $\omega\delta$ when δ is small enough.*

4 Implementation

Suppose we want to find well-conditioned real witness points of the system $p = \{p_1, ..., p_k\} \subseteq \mathbb{R}[x_1, ..., x_n]$.

To regularize the potential singularities of the given system p, we first perturb the input to yield a nearby system $f = p + \epsilon$ as shown in the previous work [37]. The optimization problem (3) can be solved by using the homotopy continuation method.

The basic idea is to choose the solutions with tiny imaginary part and project them back to the original system p. If the Jacobian of some point is near rank deficiency, then we apply the new algorithm FollowComp combined with optimal direction and step size control techniques to move the point along $V_{\mathbb{R}}(f)$ until the condition of the Jacobian is tolerable. Finally, we project this new point back $V_{\mathbb{R}}(p)$.

Since we need to do the computation in projective space for a better numerical stability, we introduce two transformations:

$$\mathcal{H} : x \in \text{ affine space } \mathbb{R}^n \to y \in \text{ projective space } \mathbb{P}^n$$

where $y_0 = 1/\sqrt{1 + \|x\|^2}$, and $y_i = x_i/\sqrt{1 + \|x\|^2}$ for $i = 1, ..., n$, and

$$\mathcal{H}^{-1} : y \in \mathbb{P}^n \to x \in \mathbb{R}^n$$

where $x_i = y_i/y_0$ for $i = 1, ..., n$.

The Newton iteration method to find a solution of a square system F with an initial approximation y is denoted by Newton(F, y).

Next we describe the algorithm FollowComp.

Algorithm FollowCurve

Input: f: a perturbed system given by $\{f_1, ..., f_k\} \subset \mathbb{R}[x_1, \ldots, x_n]$
x = an approximate solution of $f = 0$
κ = threshold used in path tracking to control smallest singular value

1. Homogenize the system f to obtain $\bar{f}$
2. Construct the system $g = \{\bar{f}, (\sum x_i^2 - 1)/2\}$ and $y = \mathcal{H}(x)$
3. Compute θ_i of g_i and construct $\tilde{g}$ by Proposition 3.4
4. Find σ_{k+1} of $\mathcal{J}_p(\tilde{g})$ by SVD, if $\sigma_{k+1} > \kappa$ then return $\mathcal{H}^{-1}(y)$
5. Let $\delta = \frac{\sigma_{k+1}}{3.2\mu}$ where $\mu = \sqrt{(k+1)(n+1)}$
6. Construct the square system $\{G, L\}$ by Equations (19) and (29)
7. Update $y = y + Hc^t\delta$ and let $y' = \text{Newton}(\{G, L\}, y)$
8. If $\|y' - y\| > 1.03\delta$ then let $\delta = \delta/2$ and go to step 7
   Else let $y = y'$ and go to step 3

Output: A well-conditioned point on $V_{\mathbb{R}}(f)$

Usually the poor conditioning region appears close to a singular component or point (e.g. at the intersection of two irreducible components). Fortunately the dimension of the singular set is lower than the dimension of regular set, the likelihood of leaving this poor conditioning region is quite large.

However, for the algorithm FollowComp, we first point out that the optimal direction technique can not guarantee its termination for any threshold κ. Since it is still a local method, it may oscillate near a local maximum of σ_{k+1} and unfortunately $\sigma_{k+1} < \kappa$. In practice, we can detect such oscillations and terminate the procedure. Then choose a new random vector in our plane-distance construction (3) to produce other points.

Secondly, we should also mention that once divergence or path-jumping is detected in step 8 of the algorithm, the step size δ will be reduced by a half and then repeat the prediction-correction step in 7. Eventually this procedure will terminate and it will converge to a point on the same component. Furthermore an upper bound of the iterations can be analyzed using the condition number techniques in [9, 8]. Thus, path-jumping can be avoided in our algorithm theoretically.

We implemented the algorithms in Maple 16 together with a Maple interface to Hom4Ps2. Although we can not verify the regularity assumption

# steps	sing. val. σ	$\mathit{dist}\,(\,\boldsymbol{z}_i, \boldsymbol{z}_{i+1}\,)$
0	0.0000102	0.112e-5
20	0.0000298	0.312e-5
40	0.0000873	0.914e-4
60	0.000256	0.0000268
80	0.000751	0.0000786
100	0.00221	0.000231
120	0.00653	0.000683
140	0.0196	0.00205
160	0.0613	0.00639
180	0.199	0.0208
197	0.372	0.041
200	0.372	0.041

Table 2: Performance of Algorithm FollowComp

in advance, it can be detected if the perturbed system has no real solutions or the Jacobian is always near rank-deficiency during path tracking.

Example 4.1 *Consider a polynomial of three variables* $f = y^2 - x^2 - x^3 + z^2$. *Homogenizing, normalizing and appending the sphere equation we get a system* $0.25\,wy^2 - 0.25\,wx^2 - 0.25\,x^3 + 0.25\,wz^2, 0.5\,x^2 + 0.5\,y^2 + 0.5\,z^2 + 0.5\,w^2 - 0.5$. *Let the initial point be* $x = 0.0000134, y = 0.00000786, z = -0.0000131, w = 0.999$ *where the smallest singular value of the Jacobian is* 0.0000102. *To see the details in the tracking, we set* κ *be a large number in* FollowCurve *and force it to run* 200 *steps. We report the smallest singular value* σ_{k+1}, *the distance* $\mathit{dist}\,(\,\boldsymbol{z}_i, \boldsymbol{z}_{i+1}\,)$ *when* i *is a multiple of* 20 *in Table 2. Note that after step* 197, *the smallest singular value oscillates near a local maximum* 0.372 *since* c *defined in Lemma 3.7 approximately belongs to* $\langle \mathcal{J} \rangle$.

Interestingly, Figure 3 strongly indicates that the cost to obtain a good point is roughly proportional to $-\log_{10}(\sigma_{k+1})$ *at the initial point for this specific example.*

Now let us attempt to explore the phenomenon shown above. We can assume the starting $\sigma_{k+1}^{(0)}$ is small enough such that each update by $\delta = \frac{\sigma_{k+1}}{3.2\mu}$ causes a tiny change in $\sqrt{cHc^t}$. Thus $\Delta\sigma = \sqrt{cHc^t}\ \delta$ implies that

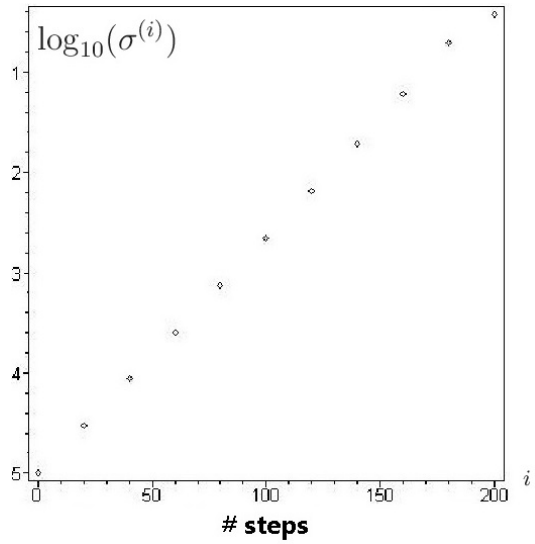

Figure 3: Plotting of $\log_{10}(\sigma^{(i)})$ against the number of steps, where $\sigma^{(i)}$ is the smallest singular value of the i-th step for Example 4.1.

$\sigma_{k+1}^{(i+1)} - \sigma_{k+1}^{(i)} = a\sigma_{k+1}^{(i)}$ where $a = \frac{\sqrt{cHc^t}}{3.2\mu}$. Suppose after m prediction-correction steps it reaches a small threshold κ, we have $(1+a)^m \sigma_{k+1}^{(0)} = \kappa$. Therefore,

$$m = \frac{\log \kappa - \log \sigma_{k+1}^{(0)}}{\log(1+a)}.$$

This empirical formula explains the linear relation between the number of steps and $\log \sigma_{k+1}$ when σ_{k+1} is small in Figure 3.

Example 4.2 *To show this typical behavior, we consider a three dimensional system* $\{f_1 \cdot f_2, f_3, f_4\}$ *in variables* $\{u, v, w, x, y, z\}$ *where* $f_1 = 88 - 40\ xw - 10\ wy$, $f_2 = -40\ z - 82\ u^2 - 82\ uy$, $f_3 = -5.1\ z - 8\ uv - 7.2\ v^2 + 3.6\ zw + 7.1\ xz + 5.4\ yz$, $f_4 = -1.8\ x - 4.5\ vw - 3\ vy + 7.3\ zw - 3.7\ xy + 8.3\ yz$. *Suppose we have a point* p *at* $\{u = .2206606267855835, v = 2.062128087890764, w = -.6237288386440759, x = -4.553242196956194, y = 4.104273703065960, z = -1.956402576062135\}$ *which is close to the singular variety* $\{f_1 = f_2 = f_3 = f_4 = 0\}$ *with the smallest singular value* 0.11107×10^{-10}.

To move p *out of the ill-conditioned region, we apply the algorithm and after* 18 *steps the new point becomes*

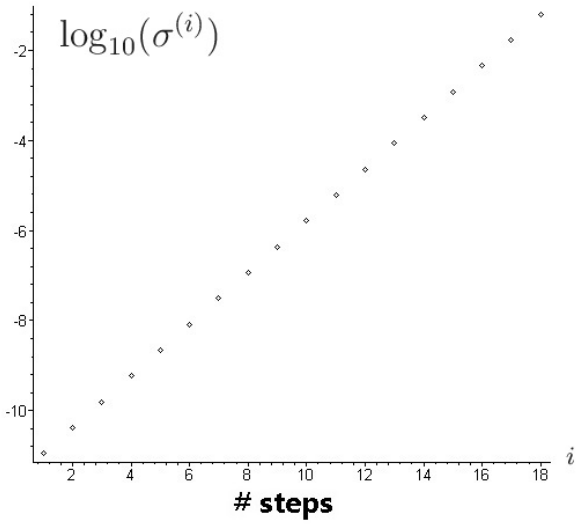

Figure 4: Plotting of $\log_{10}(\sigma^{(i)})$ against the number of steps, where $\sigma^{(i)}$ is the smallest singular value of the i-th step for Example 4.2.

$\{u = .22651599665957, v = 2.0600058541281, w = -.62359532321758, x = -4.55391828272596, y = 4.10395729152011, z = -1.95742849126669\}$ *with the smallest singular value* 0.2376.

We can also observe this linear relation between $\log_{10}(\sigma)$ *and the number of steps in Figure 4.*

5 Conclusions

In this paper we give a preliminary analysis of the complexity and the numerical stability of our critical point methods. Our experiments applied these methods to the problem of finding at least one well conditioned witness point on each connected components of a real variety. Our methods use perturbation, path tracking and projection techniques together with an optimal-direction strategy and an adaptive step size control strategy.

A numerical algorithm combining these strategies is given. The experimental results show that it is successful on our test problems for finding real witness points without path-jumping. As by-products, our approach can also be used in numerical algebraic geometry for computing well-conditioned

complex witness points.

Future research includes loosening the assumptions A_1 and A_2, to address more degenerate cases. Such research includes methods for deflating higher multiplicity components, and also addressing problems with equations which are sums of squares. We will apply the new strategies to other methods e.g. the closest method to ours, that of Hauenstein [18].

Future work includes classifying real witness points, i.e. determining whether a pair of points belong to the same connected component. Another future direction is to apply our methods to the computation of roadmaps of semi-algebraic sets [17, 3, 31, 32, 4].

In the paper we assume the local convergence of Newton iteration. The certified numerical homotopy tracking by Beltran and Leykin [8] applied to systems of polynomials with integer coefficients. An interesting research problem is to combine their results with our approach to input polynomials with integer or rational coefficients.

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