Finding Points on Real Solution Components and Applications to Differential Polynomial Systems

Wenyuan Wu*
Chongqing Inst. of Green and Intelligent Techn.
Chinese Academy of Sciences
wuwenyuan@cigit.ac.cn

Greg Reid†
Applied Mathematics Dept.
Western University, Canada
reid@uwo.ca

ABSTRACT

In this paper we extend complex homotopy methods to finding witness points on the irreducible components of real varieties. In particular we construct such witness points as the isolated real solutions of a constrained optimization problem.

First a random hyperplane characterized by its random normal vector is chosen. Witness points are computed by a polyhedral homotopy method. Some of them are at the intersection of this hyperplane with the components. Other witness points are the local critical points of the distance from the plane to components. A method is also given for constructing regular witness points on components, when the critical points are singular.

The method is applicable to systems satisfying certain regularity conditions. Illustrative examples are given. We show that the method can be used in the consistent initialization phase of a popular method due to Pryce and Pantelides for preprocessing differential algebraic equations for numerical solution.

Categories and Subject Descriptors: G.1.8 **General Terms:** algorithms, design

Keywords: numerical algebraic geometry, real algebraic geometry, homotopy continuation, singular critical points, witness points, differential algebraic equations.

*The work is partly supported by the projects NSFC-11001040, West Light Foundation of CAS from China (Y21Z010C10).

†Work partly funded by Reid's NSERC grant (government of Canada).

ISSAC'13, June 26–29, 2013, Boston, Massachusetts, USA.

1. INTRODUCTION

This article is a contribution to the development of numerical algorithms for real computational algebraic geometry, and the extension of such methods to systems of real differential polynomials.

It is motivated by recent progress and numerical algorithms for complex algebraic geometry, and in particular the blossoming area of numerical algebraic geometry pioneered by Sommese, Wampler, Verschelde and others. See the book [22] and also [2] for references. Recent progress in extending such methods to the real case are by Lu [17] and Besana et al. [3]. Of particular interest and closest to this paper is the recent work of Hauenstein [11]. Our work is also motivated by progress in symbolic methods for determining features of real solutions of general real polynomial systems (e.g. see [19]).

In this paper we give a numerical method for computing solution (witness) points on each real connected component for real polynomial systems satisfying certain regularity conditions. Given a random vector $\mathbf{n}$ the method computes local critical points of the distance of a hyperplane to the component in the direction $\mathbf{n}$. A method for regularizing singular critical points is also given. This method takes advantage of the availability of efficient homotopy solvers which exploit sparsity and structure of the polynomial system [14].

The real solving method we describe in this paper is applied to the consistent initialization step of the Pryce-Pantelides method [25, 18] for preprocessing differential algebraic equations (DAE) for numerical solution. The Pryce-Pantelides method is successful under certain regularity conditions (in particular that certain system Jacobians are nonsingular). Though not guaranteed the success rate is high enough that it has been implemented in a number of problem solving environments. For example it is the first method of choice in Maple's DAE solving environment MapleSim (see [25] for references).

The Pryce-Pantelides method takes as input a square system of DAE (i.e. the number of equations and unknowns are equal) and consists of a differentiation (prolongation) step and a consistent initialization step.

The prolongation step involves solving an optimization problem. The results are used to determine which higher order derivatives of equations in the system should be appended to the original DAE system, to form an prolonged system of DAE, that implicitly includes its missing constraints. For background references and interpretation in terms of differential elimination theory see our paper [25]. For example as shown in [25] the optimization corresponds to the choice of a partial ranking minimizing a differential Hilbert function of the DAE.

The consistent initialization step requires the determination of initial conditions lying on the variety of the DAE in the space where its unknowns and derivatives are regarded as indeterminates. We will apply our real solving method to this step, and exploit the fact that the required regularity conditions are exactly the same as those needed by Pryce-Pantelides. When these steps are successful standard index one numerical DAE solvers can be applied to the output. Indeed there is even a complexity analysis available showing the low (polynomial) cost of this numerical solution [13].

1.1 Numerical algebraic geometry

The methods of numerical algebraic geometry compute approximate complex points on all irreducible solution components of multivariate complex systems of polynomial equations. Such witness points on components of each possible dimension are obtained by slicing with random planes of equal co-dimension. The points are computed with efficient homotopy methods.

Consider the set $\mathbb{C}[x_1, x_2, ..., x_n]$ of multivariate polynomials with complex coefficients in the complex variables $x = (x_1, x_2, ..., x_n) \in \mathbb{C}^n$. Then $\mathbb{C}[x_1, x_2, ..., x_n]$ is a ring and a system of m polynomials $p_1(x)$, $p_2(x)$, ... , $p_m(x)$ in $\mathbb{C}[x_1, x_2, ..., x_n]$ yields a system of m multi-variate polynomial equations $p(x) = (p_1(x), p_2(x), ..., p_m(x)) = 0$. Its solution set or variety is $V(p) = \{x \in \mathbb{C}^n : p(x) = 0\}$. The regular points of $V(p)$, or $\mathrm{reg}(V(p))$, are points at which $V(p)$ is a local complex Euclidean manifold of possible dimension: 0 (points), 1 (curves), ... , $n-1$ (hyper-surfaces). Then $\mathrm{reg}(V(p))$ can be partitioned into a disjoint union of subsets, under the equivalence relation of connectedness. Finally closure in the Euclidean topology of these sets yields the irreducible components of a complex polynomial system.

1.2 Real algebraic geometry

Finding real solution components, is usually the case of interest in applications. Naive extension of the complex approach to the real case fails, since such random planes may not intersect some (e.g. compact) components.

Suppose that $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and consider a system of k multivariate polynomials $p_1(x)$, $p_2(x)$, ... , $p_k(x)$ in the polynomial ring $\mathbb{R}[x_1, x_2, ..., x_n]$. Its solution set or variety is

$$V_{\mathbb{R}}(p_1, ..., p_k) = \{x \in \mathbb{R}^n : p_j(x) = 0,\ 1 \le j \le k\} \quad (1)$$

Real algebraic geometry is a vast subject with many applications. For a modern text with many references on computational real algebraic geometry see [1].

Sturm's ancient method on counting real roots of a polynomial in an interval is central to Tarski's real quantifier elimination [23] and was further developed by Seidenberg [21]. One of the most important algorithms of real algebraic geometry is cylindrical algebraic decomposition. CAD was introduced by Collins [7] and improved by Hong [12] who made Tarski's quantifier elimination algorithmic. This algorithm decomposes $\mathbb{R}^n$ into cells on which each polynomial of a given system has constant sign. The projections of two cells in $\mathbb{R}^n$ to $\mathbb{R}^k$ with $k < n$ either don't intersect or are equal. The double exponential cost of this algorithm [8], is a major barrier to its application. See [6] and [5] for modern improvements using triangular decompositions.

A paper for obtaining witness points for the real positive dimensional case, closely related to our approach, is [19] (also see [9]). Homotopy methods are used in [17] and [3] for real algebraic geometry. Lasserre et al [15] uses semi-definite programming and interestingly that approach is related to the prolongation-projection method used in geometrical completion of differential systems (also see Wu and Zhi [26]).

2. COMPUTING WITNESS POINTS ON REAL COMPONENTS

We consider systems

$$f = (f_1, f_2, ..., f_k) = 0 \quad (2)$$

of k polynomials from $\mathbb{R}[x_1, ..., x_n]$ which satisfy the following assumptions:

A_1: $V_{\mathbb{R}}(f_1, f_2, ..., f_i)$ has dimension $n - i$ for $1 \le i \le k$.

A_2: the ideal $I_i = \langle f_1, f_2, ..., f_i \rangle$ is radical for $1 \le i \le k$.

These assumptions mean that the Jacobian of the system $\{f_1, f_2, ..., f_i\}$ has full row rank at generic points of $V_{\mathbb{R}}(f_1, f_2, ..., f_i)$ for $1 \le i \le k$. If a system f satisfies these two assumptions, we say f is *regular*.

The application of the Pryce-Pantelides method to DAE in Section 4 requires the assumptions A_1 and A_2. Implementations and many successful applications of this method in problem solving environments such as MapleSim, Mathematica, gPROMS, Modelica and EMSO attest to these assumptions being often satisfied in practise.

In the longer term we aim to also develop methods that don't require such assumptions. However they lead to very efficient and fast algorithms. Our approach is to treat well-conditioned problems first before addressing more singular and ill-conditioned problems.

We outline the main steps of our method, postponing details, such as singular cases to later sections.

To begin with, we choose a random point $\hat{x}$ and a random vector $\mathbf{n}$ in $\mathbb{R}^n$. Consider the random hyperplane H in $\mathbb{R}^n$ through $\hat{x}$ with normal $\mathbf{n}$.

For illustration of some of the main ideas we use a simple example in $\mathbb{R}^2$ which has variety given in Figure 1. Many methods are known for this case, and also for the case of real hyper-surfaces in $\mathbb{R}^3$. The novelty of our methods, is primarily for the case of $k > 1$ polynomials in $\mathbb{R}^n$, defining equi-dimensional varieties of dimension $n-k$. Our use of the co-dimension one case and Example 2.1 is purely illustrative.

EXAMPLE 2.1. *Figure 1 displays the variety of a single polynomial $\mathbb{R}^2$ given by:*

$$\begin{aligned} f \;=\; & \left(x^2+y^2-1\right)\cdot\left(x-3-y^2\right)\cdot\left(x+(y+2)^3\right) \\ & \cdot\left(\left(2\,y-4\right)^2-\left(2\,x-x^2\right)^3\right) \;=\; 0 \end{aligned}$$

First we compute the intersection of f and a random line H, that lie distance 0 from the variety, and obtain the point P_1. Note that the resulting system $f = 0$ together with $H = 0$ is one dimension less, and belongs to the case of $k+1$ polynomials. Since we have intersected with a random hyperplane the resulting system of $k+1$ polynomials will also satisfy the regularity assumptions A_1 and A_2.

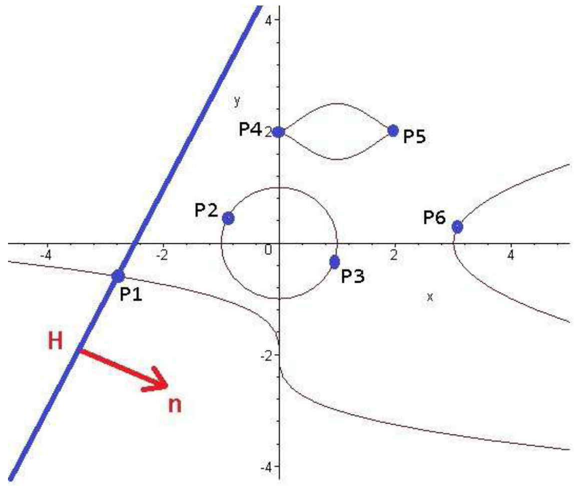

Figure 1: H is a random line through the random point $\hat{x}$ with random normal n. The variety consists of 4 one dimensional components. Point P_1 is the only point at which the variety intersects H at distance 0 from H. The regular local critical points of the normal distance from H are given by P_2, P_3, P_6. The singular critical points are P_4 and P_5.

In the simple case in $\mathbb{R}$ above it would amount to finding the isolated zeros of a zero dimensional system defined by H and $f = 0$.

The closure of any component that does not intersect H must (with probability one) contain critical points of the normal distance to H. If not, then the component would become asymptotically close to H, without touching H. However the randomness of the defining normal of H, and finiteness of the number of such asymptotes, implies that this can only happen on a set of lower measure.

Returning to our illustrative example, it shows the critical points P_2, P_3, P_4, P_5 and P_6 on their corresponding varieties. The above description is essentially a geometric one. However any algorithm that computes it, must use the equations defining the variety and results in well-known and nontrivial difficulties for higher multiplicities and overdetermined systems. Thus we invoke the assumptions A_1 and A_2 to yield enough regularity so that our algorithms are well-conditioned, and the encountered Jacobians have full rank.

Regular critical points (e.g. P_2, P_3, P_6) will have $\lambda \nabla f = \mathbf{n}$ in $\mathbb{R}^2$. Generally, in $\mathbb{R}^n$

$$\mathbf{n} \in \operatorname{span}\{\nabla f\} = \operatorname{span}\{\nabla f_1, \cdots, \nabla f_k\} \tag{3}$$

Consequently these critical points are the solutions of the Lagrange optimization problem

$$f = 0, \qquad \sum_{i=1}^{k} \lambda_i \nabla f_i = \mathbf{n} \tag{4}$$

Here $\mathbf{n}$ is a random vector in $\mathbb{R}^n$ and (4) has $n+k$ equations and $n+k$ unknowns $(x, \lambda) = (x_1, ..., x_n, \lambda_1, ..., \lambda_k)$.

Remark 2.1. *The point-distance formulation to obtain real points as the critical points of the distance from a component to a random point can be found in [19, 11]. The main difference between that previous work and our paper is that we find critical points of the distance to a random hyperplane rather than to a random point. If the random point in the point-distance formulation is far away from a component, the corresponding system may have poor condition number. However, in our plane-distance approach the random hyperplane or random normal vector is invariant under translation. So the distance to the component does not affect the conditioning. This usually leads to a square system with lower mixed volume the solution of which is more numerically stable and efficient.*

2.1 The case of regular critical points

We first consider the case of regular local critical points. The singular cases will be considered in Section 2.2. For the regular case we use homotopy continuation methods to solve the new system and find the real critical points which are real. Homotopy continuation methods can determine all isolated complex roots. To apply the methods, we need to show the critical points are isolated in complex space.

Proposition 2.2. *Suppose (x^0, λ^0) is a real solution of (4) with random vector $\mathbf{n}$ and also that the Jacobian of f at this point is of full row rank. Then (x^0, λ^0) is an isolated root of (4) in $\mathbb{C}^{n+k}$ with probability 1.*

Proof. Since the Jacobian of f at (x^0, λ^0) has full row rank k, by the implicit function theorem [10] there are k variables that can be locally expressed as smooth functions of the other variables. Without loss of generality, we write $x_1 = x_1(z), ..., x_k = x_k(z)$, where $z = (x_{k+1}, ..., x_n)$. Substituting into the second equation of (4), that is into $\sum_{i=1}^{k} \lambda_i \nabla f_i = \mathbf{n}$, yields a square system of n equations, which we denote by $g(z, \lambda) = \mathbf{n}$.

Thus, g is a smooth mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. By Sard's Theorem [20], [10], for almost all $\mathbf{n}$, the Jacobian of g at (z^0, λ^0) has rank n. Consequently, the Jacobian of the full system of $n+k$ equations (4) has full rank $n+k$ at $(x^0, \lambda^0) \in \mathbb{R}^{n+k}$. The rank is the same at this point in $\mathbb{C}^{n+k}$. So (x^0, λ^0) is an isolated root in $\mathbb{C}^{n+k}$. $\square$

Example 2.2. *Let $f = x^2 + y^2 - 1$. Let H be the line through $\hat{x} = (0, -3)$ with $\mathbf{n} = (2, -1)^t$, that is the line $y = 2x - 3$. Then (4) is*

$$f = x^2 + y^2 - 1 = 0\,, \qquad \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{5}$$

Simplifying the system, we have $x^2 + y^2 - 1 = 0, x = -2y$. This leads to two real roots: $P_1 = (-2\sqrt{5}/5, \sqrt{5}/5)$ and $P_2 = (2\sqrt{5}/5, -\sqrt{5}/5)$ corresponding to the minimum distance point and the maximum distance point respectively as shown in Figure 2.

Here f satisfies the full rank Jacobian condition and the critical points are regular. In Example 2.3 however the critical points on a connected component are singular.

Example 2.3. *Let $f = y^2 - (x - x^2)^3$ with graph shown in Figure 3 and consider H given by $y = kx + b$.*

Consider the case when the absolute value of the slope of H, i.e. $|k|$, is larger than the maximum slope of the tangent line to the curve $y^2 - (x - x^2)^3 = 0$. Then when H is $y = 2x + 3$, the critical points are the singular points $(0, 0)$ and $(1, 0)$.

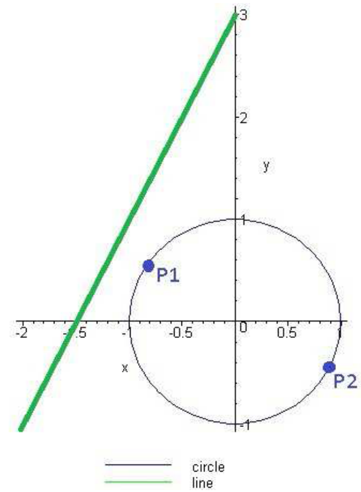

Figure 2: The circle $x^2 + y^2 = 1$ and the line $y = 2x + 3$. The normal is $(2, -1)^t$ and the critical points of the normal distance from the line to the circle are displayed as P_1 and P_2.

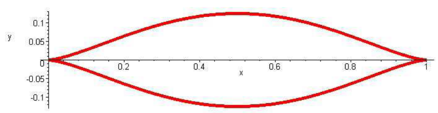

Figure 3: $y^2 - (x - x^2)^3 = 0$

Then the corresponding optimization problem

$$f = y^2 - (x - x^2)^3 = 0, \quad \lambda \begin{pmatrix} -3(x - x^2)^2(1 - 2\,x) \\ 2y \end{pmatrix} = \begin{pmatrix} -k \\ 1 \end{pmatrix} \tag{6}$$

has no real solutions. In fact, the probability of hitting the singular points can be quite large. We illustrate this by calculating this probability for the curve $t\,y^2 - (x - x^2)^3 = 0$ *where* t *is a parameter. For the solution branch,* $y = (x - x^2)^{3/2}/\sqrt{t}$ *we have*

$$\frac{d^2y}{dx^2} = \frac{3}{4}\,\frac{(1 - 8\,x + 8\,x^2)}{\sqrt{x\,(1 - x)}\sqrt{t}}.$$

So when $x = \frac{2 \pm \sqrt{2}}{4}$ *the slope of the curve* $\frac{dy}{dx}$ *attains the critical values* $\frac{\mp 3}{8\,\sqrt{t}}$*. Thus (6) has no real solutions if and only if* $k \in (-\infty, -\frac{8}{3}\sqrt{t}] \cup [\frac{8}{3}\sqrt{t}, \infty)$*.*

Let H *have normal* $\boldsymbol{n} = (sin(\theta), cos(\theta))$*. Assume the random variable* θ *is uniformly distributed in* $[0, 2\pi]$*. Then the probability of hitting the singular points* $(0, 0)$ *and* $(1, 0)$ *is* $\frac{2}{\pi}\arctan(\frac{8}{3}\sqrt{t})$*. When* $t = 1$*, the probability is* 0.772 *and if* $t = 10$ *the probability increases to* 0.925*. This fact indicates that avoiding hitting singular points can be difficult for such curves.*

2.2 Singular critical point

As is usual in numerical investigations, we first considered the regular cases before considering singular cases. In this section we consider the case illustrated in Figure 1, where some of the critical points are singular.

The key is to study the local properties of a perturbed system. We use the following notation. For any point $p \in \mathbb{R}^n$, $B_p(r) = \{x \in \mathbb{R}^n \mid \|x - p\| \le r\}$. If $p \in V_{\mathbb{R}}$, we define $N_p = V_{\mathbb{R}} \cap B_p$.

Let f be a regular system of k equations and consider a regular point $p \in V_{\mathbb{R}}(f)$. By appending linear equations $L(x) = L(p)$ to f, where L is orthogonal to the Jacobian of f at p, we obtain a square system:

$$G = \begin{pmatrix} f \\ L \end{pmatrix} \tag{7}$$

with $\mathcal{J} = \frac{\partial G}{\partial x}$. A perturbation $\epsilon \in \mathbb{R}^k$ is added to f to yield the perturbed system

$$\tilde{G} = \begin{pmatrix} f + \epsilon \\ L \end{pmatrix} \tag{8}$$

The variety of a perturbed system with perturbation $\epsilon \in \mathbb{R}^k$ is denoted by $V_{\mathbb{R}}^{\epsilon}$. Similarly $N_p^{\epsilon} = V_{\mathbb{R}}^{\epsilon} \cap B_p$.

Consider the linear homotopy connecting these two systems given by $H(x, t) = t\tilde{G}(x) + (1 - t)G(x) = 0$. We follow a curve $x(t)$ starting from $x(0) = p$ satisfying $H(x(t), t) = 0$ by solving a differential algebraic equation:

$$\frac{dH}{dt} = \frac{d}{dt}\begin{pmatrix} f + t\epsilon \\ L \end{pmatrix} = \mathcal{J} \cdot \frac{dx}{dt} + \begin{pmatrix} \epsilon \\ 0 \end{pmatrix} = 0 \tag{9}$$

with initial condition $x(0) = p$. The solution $q = x(1)$ of (9) is uniquely determined if the perturbation is small enough so that the Jacobian $\mathcal{J}$ is invertible for $t \le 1$.

PROPOSITION 2.3. *Suppose* $\mathcal{J}$ *is invertible in* $B_p(r)$ *for some* $r > 0$*. Let* $\delta = \max\{\|\mathcal{J}^{-1}(x)\| \mid x \in B_p(r)\}$*. If* $\|\epsilon\| < r/\delta$*, then* $\|p - q\| \le \|\epsilon\|\delta$*. Moreover, the local dimension of the perturbed variety at* q *is* $n - k$ *and* $dist(p, N_p^{\epsilon}) \le \|\epsilon\|\delta$*.*

PROOF. Since

$$\left\| \int_0^1 \mathcal{J}^{-1} \cdot \begin{pmatrix} \epsilon \\ 0 \end{pmatrix} dt \right\| \le \|\epsilon\| \cdot \int_0^1 \|\mathcal{J}^{-1}\|\, dt \le \|\epsilon\|\delta$$

the distance between p and q is at most $\|\epsilon\|\delta$. Thus $dist(p, N_p^{\epsilon}) \le dist(p, q) \le \|\epsilon\|\delta$. Because $\mathcal{J}$ is invertible in $B_p(r)$, the Jacobian of $\tilde{G}$ is also invertible at q. Thus the local dimension of $f + \epsilon$ at q is $n - k$. □

THEOREM 2.4 (REGULARIZATION THEOREM). *If* $f = \{f_1, ..., f_k\}$ *is regular and* c *is sufficiently small, then for almost all* $\|\epsilon\| < c$*,* $f + \epsilon$ *has dimension* $n - k$ *and there are no singular points on* $V_{\mathbb{R}}(f + \epsilon)$*.*

PROOF. Consider f as a smooth mapping from $\mathbb{R}^n$ to $\mathbb{R}^k$ with $k < n$. The image of the critical set, denoted by S has Lebesgue measure zero in $\mathbb{R}^k$ by Sard's theorem.

By Proposition 2.3, when the perturbation is small enough, $V_{\mathbb{R}}(f + \epsilon) \ne \emptyset$. Thus, for almost all $\|\epsilon\| < c$, $f + \epsilon$ has no singular points and $\dim V_{\mathbb{R}}(f + \epsilon) = n - k$. □

EXAMPLE 2.4. *Consider* $f = \{y^2 + z^2 - (2\,x - x^2)^3, z - y^2\}$*. Figure 4 clearly shows a singular point at the origin. To show the intersection explicitly, we solve* y *and* z *in terms of* x *obtaining two solution branches:*

$$y = \sqrt{z}, z = (-1 + \sqrt{1 - 32\,x^3 - 48\,x^4 + 24\,x^5 - 4\,x^6})/2$$

and

$$y = -\sqrt{z}, z = (-1 + \sqrt{1 - 32\,x^3 - 48\,x^4 + 24\,x^5 - 4\,x^6})/2.$$

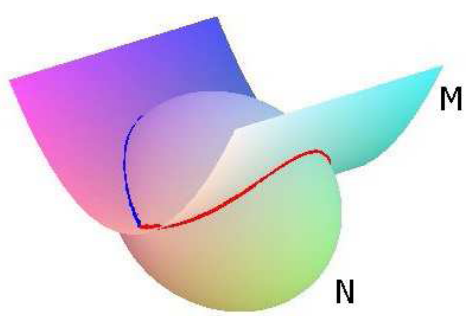

Figure 4: Two surfaces M and N and their intersection. Here M: $z-y^2=0$ and N: $y^2+z^2-(2x-x^2)^3=0$.

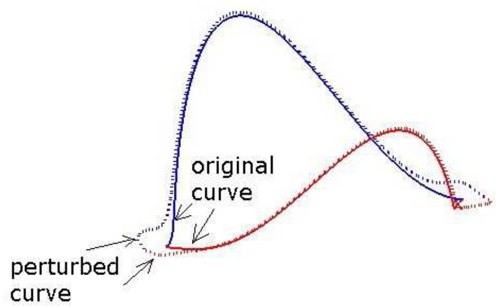

Figure 5: Perturbed and original real varieties

Consider the perturbed system $f+\epsilon=\{y^2+z^2-(2\,x-x^2)^3+\epsilon_1, z-y^2+\epsilon_2\}$. *Consequently,*

$$z=(-1+\sqrt{1-32\,x^3-48\,x^4+24\,x^5-4\,x^6-4\epsilon_1-4\epsilon_2})/2.$$

Suppose $\epsilon_1=-0.04, \epsilon_2=0.015$. *The perturbed and unperturbed curves are shown in Figure 5.*

Set $\mathbf{n}=(-1,0,0.2)$ *in (4). We solve the system* $\{f+\epsilon=0, \lambda\cdot\nabla(f+\epsilon)=\mathbf{n}\}$ *by* `Hom4Ps2` *and obtain* 26 *complex roots, including the following two real roots*

$$(-0.108289277,0,0.01), \quad (2.108289277,0,0.01).$$

Let us consider $p=(-0.108289277,0,0.01)$. *To address the numerical difficulties encountered near the singularity, we project onto the original curve. After* 20 *Newton iterations, we obtain the point* $(-0.000179,0,0)$. *The smallest singular value of the Jacobian at this point is* 0.769×10^{-6}.

One possible way to overcome the numerical difficulty is to solve $G=\{f=0,\lambda\cdot\nabla f=\mathbf{n}\}$ *by using deflation methods [16]. However it is easily seen that the critical point* $(0,0,0)$ *does not satisfy* G.

Here we present alternative way to avoid the numerically difficult region. Firstly, we move p *along the perturbed curve to another point* $q=(0.101,-0.135,0.0283)$ *where the singular values of the Jacobian are* $(1.04,0.34)$. *Secondly, we project the point* q *to the original curve and find* $q'=(q'_1,q'_2,q'_3)$ *where* $q'_1=0.122267538554529$, $q'_2=-0.109354334436423$ *and* $q'_3=0.0119583704597381$. *At* q' *the residue is* $(0.59\times10^{-12},-0.29\times10^{-12})$ *and the singular values are* $(1.02,0.353)$. *So* q' *is a regular point on the target curve with high accuracy. Now we use this example to verify Proposition 2.3. By the definition* $\delta\le1/0.34$ *and* $\|\epsilon\|=0.0427$, *and we have* $dist(q,q')=0.0372<\|\epsilon\|\delta$.

REMARK 2.5. *In contrast to the approach in Hauenstein [11], our critical points are defined differently, ie. by distance to a hyperplane, rather than distance to a point. In [11] the author applied an endgame and adaptive precision tracking technique to deal with singular cases. Such cases can occur with high probability especially when the components have "cusps". In the case where there are nearby smooth real points, we give a regularization method for the singular critical points without extending the hardware precision.*

3. IMPLEMENTATION

To address the potential singularities of a given system f, we first perturb the input to yield a nearby system $\tilde{F}$. Then a random linear equation $L=\mathbf{n}\cdot x+1$ is defined. There are two systems to analyze:
(1) System $G=\{\tilde{F},\lambda\cdot\nabla F=\mathbf{n}\}$ (2) System $\{\tilde{F},L\}$.

We solve the square system $\tilde{G}$ of (8) in $\mathbb{C}$ by a Homotopy continuation package, e.g. HOM4PS2 by T.Y. Li et al [14]. We choose only the real roots and discard any imaginary roots. For each real point, we need to project to the variety $V_{\mathbb{R}}(F)$ by the algorithm PROJ2MANIFOLD described below. If the Jacobian is near rank deficiency, then we apply the algorithm FOLLOWCURVE to move the point along $V_{\mathbb{R}}(\tilde{F})$ until the condition of the Jacobian is tolerable. Finally, we compute the projection from this new point onto $V_{\mathbb{R}}(F)$.

For the second system of lower dimension, we can consider it as a new input and solve it recursively by the method introduced above.

Now we describe our algorithms.

Algorithm PROJ2MANIFOLD
Input: System $f=\{f_1,...,f_k\}\subset\mathbb{R}[x_1,\ldots,x_n]$, point p

- Solve N the right null space of $\mathcal{J}_p(F)$ by SVD
- Construct the linear system $L:\ N\cdot x=N\cdot p$
- Let $\bar{F}=\{F,L\}$ which is a square system
- Apply Newton iteration to p to yield q

Output: A regular point q on $V_{\mathbb{R}}(f)$

Algorithm FOLLOWCURVE
Input: $f+\epsilon$: a perturbed system given by $\{f_1+\epsilon_1,...,f_k+\epsilon_k\}\subset\mathbb{R}[x_1,\ldots,x_n]$
$p=$ an approximate solution of $f+\epsilon=0$
$K=$ control condition ♯ used in curve tracking

- Solve N the right null space of $\mathcal{J}_p(f+\epsilon)$ by SVD
- Construct an $(n-k-1)\times n$ linear system $L:\ A\cdot x=A\cdot p$, where A consists of the last $n-k-1$ vectors of N to produce a curve to follow to a regular point.
- Let $\bar{F}=\{f+\epsilon,L\}$ which is $(n-1)\times n$ system and $\bar{F}(p)=0$
- Track the curve from p to q by prediction-projection method until $\|\mathcal{J}_q^+\|<K$, where $\mathcal{J}^+$ is the pseudo-inverse of $\mathcal{J}$.

Output: A regular point q on $V_{\mathbb{R}}(f+\epsilon)$

Algorithm RealWitnessPoint

Input: $f = \{f_1, ..., f_k\} \subset \mathbb{R}[x_1, \ldots, x_n]$ satisfying regularity assumptions A_1 and A_2

c = upper bound on the perturbation

K = control condition $\sharp$ used in curve tracking

- Choose a random real hyperplane $H: \mathbf{n} \cdot x = 1$
- Construct $G = \{f + \epsilon, \lambda \cdot \nabla f = \mathbf{n}\}$, where $\epsilon \in \mathbb{R}^k$ and $\|\epsilon\| < c$
- Let S be the set of real roots of $G = 0$ by Hom4Ps2
- For each point $p \in S$,
  - if $\|\mathcal{J}_p^+\| > K$ (i.e. condition poor) then
    $q = \text{FollowCurve}(f + \epsilon, p, K)$
    $q' = \text{Project2Manifold}(f, q)$
  - else $q' = \text{Project2Manifold}(f, p)$
- replace p by q' in S

Output: The real witness points of $f = 0$:
$S \cup \text{RealWitnessPoints}(\{f, H\})$

Usually the poor conditioning region where $\|\mathcal{J}_p^+\| > K$ appears close to a singular component or point (e.g. at the intersection of two irreducible components). Since the dimension of the singular set is lower than the dimension of regular set, the likelihood of leaving this poor conditioning region is quite large.

We implemented the algorithms in Maple 16 together with a Maple interface to Hom4Ps2. Although we can not verify the regularity assumption in advance, it can be detected if the perturbed system has no real solutions or the Jacobian is always near rank-deficiency during path tracking.

4. APPLICATION TO DAE

In this section we show how our real solving method can be applied to the consistent initialization of DAE in the Pryce-Pantelides method.

As shown in [25] that method is equivalent implicitly to a Riquier Basis (an object which could be computed for exact input by symbolic differential elimination algorithms). In this section we consider a crane control example. We compare the application of symbolic differential elimination with Pryce-Pantelides (coupled with our real solving method).

If successful it is very efficient since the prolongation step can be solved in polynomial time and an efficient polynomial cost method can be used to numerically solve the prolonged DAE [13]. The reader may wish to look ahead at the Table 1 below, where Pryce's method partnered by our real-solving method is much more efficient than a standard differential elimination method on a class of DAE. The reader should view this comparison cautiously as symbolic differential elimination algorithms yield more theoretically complete results, since they follow cases (and a radical differential membership result is available). Also they can apply to over-determined DAE and systems with multiplicities.

In fact as shown in our paper [25] the Pryce-Pantelides method produces an implicit form of a Riquier Basis without making potentially costly symbolic eliminations. Also the nonsingular Jacobians are precisely the conditions for the implicit function theorem, to transform Pryce's system into a Riquier Basis (see Theorem 6.12 in [25]). Those conditions are equivalent to A_1 and A_2.

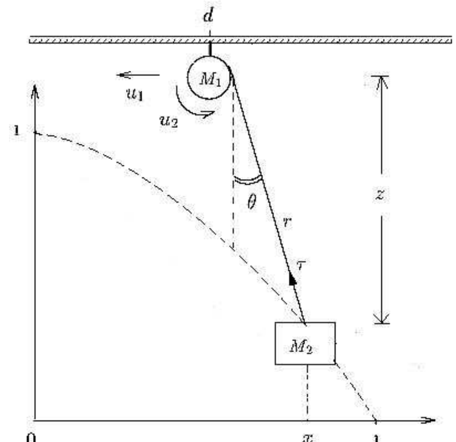

Figure 6: Control of a crane

Example 4.1 (Index 5 DAE for a crane). *This model which is illustrated in Figure 6, is discussed in [25]. The problem is to determine the horizontal velocity $u_1(t)$ and the angular velocity $u_2(t)$ of a winch of mass M_1, so that the attached load M_2 moves along a prescribed path – the dashed curve in Figure 6.*

The equations of motion are given by Visconti [24] with unknowns $\{x, x', z, z', d, d', r, r', \theta, \tau, u_1, u_2\}$:

$$x_t - x' = 0, \quad z_t - z' = 0, \quad d_t - d' = 0, \quad r_t - r' = 0$$
$$M_2\, x'_t + \tau \sin(\theta) = 0, \quad M_1\, d'_t + C_1\, d_t - u_1 - \tau \sin(\theta) = 0$$
$$M_2\, z'_t + \tau \cos(\theta) - mg = 0, \quad J\, r'_t + C_2\, r_t + C_3\, u_2 - C_3{}^2 \tau = 0$$
$$r \sin(\theta) + d - x = 0, \quad r \cos(\theta) - z = 0$$
$$H_1(x, z, t) = 0, \quad H_2(x, z, t) = 0.$$

The prescribed path of the mass M_2 is described by an algebraic equations $\{H_1 = 0, H_2 = 0\}$. The winch has moment of inertia J and is attached with a cable of length $r(t)$, making an angle $\theta(t)$ to the vertical. Substituting $sin(\theta) = s(t)$ and $cos(\theta) = c(t)$ and appending $s(t)^2 + c(t)^2 = 1$ converts the DAE to a system of differential polynomials. Applying the Pryce-Pantelides method [25], we obtain 13 ODE and 39 algebraic constraints.

To illustrate how to find a real initial point for this DAE, we reuse the polynomial system of Example 2.4 with $\{H_1 = z(t)^2 - t = 0,\ H_2 = z(t)^2 + t^2 - \left(2\, x(t) - x(t)^2\right)^3 = 0\}$. Then the total Bézout degree of the constraints becomes 21233664, but it has a block triangular structure enabling its solutions by bottom up substitution.

Choosing the initial time t randomly as in [25], say $t = 4$ and applying the homotopy method Hom4Ps2 yields 24 solutions. But all of them are complex. Since the real variety of the bottom block $H_1 = H_2 = 0$ is a bounded curve in the (x, z, t)-space as shown in Figure 5, it leads to a large chance of missing the curve when we apply a random real slicing used in our previous paper [25].

The method we have presented in the current paper, however, can find real initial points for this example, with good condition as we explained already in Example 2.4. Further the Block structure of the system, enables us to easy verify that all the relevant Jacobians for the success of the method are non-singular, by efficient bottom up substitution. Equivalently the variety defined by the DAE satisfies our assumptions A_1 and A_2.

j	diff-elim (sec) (rifsimp)	fast prolongation + real solving (sec)	# real points
1	0.31	0.063+0.12	4
2	0.69	0.063+0.22	16
3	2.39	0.063+0.17	10
4	22.48	0.063+0.39	12
5	> 3 hr	0.063+0.28	10

Table 1: Times for the crane problem by symbolic differential elimination (rifsimp) and the Pryce method. Here $H_1 = x(t)^j + z(t)^j - x(t)z(t) - t - j, H_2 = z(t)^j + x(t)z(t) - t^j + j$ for $j = 1, 2, 3, 4, 5$. Exsecuted in Maple16 on a PC under Windows 7 with 4GB of RAM, I5 cpu at 2.5 GHz.

Moreover, the computational difficulty of this problem for the symbolic differential elimination algorithm Rifsimp explosively increases with the degree d of H_1, H_2 in comparison with the numerical method as shown in the Table 1.

5. CONCLUSIONS

In this paper we give a numerical method for computing witness points on real connected components for real polynomial systems satisfying the regularity assumptions A_1 and A_2. Given a random vector $\mathbf{n}$ the method computes local critical points of the distance of a hyperplane to the component in the direction $\mathbf{n}$. A method for regularizing singular critical points is also given.

This method takes advantage of the availability of efficient homotopy solvers which exploit sparsity and structure of the polynomial system to potentially significantly reduce the number of paths following in homotopy solving. The method is pleasingly parallelisable, since homotopy paths can be followed independently on different processors. Once a witness point is determined, additional points on the component, can be further generated by other homotopies.

We demonstrated the usefulness of our plane-distance method in the consistent initialization step of the Pryce-Pantelides method. In particular its regularity conditions are the same as those of Pryce-Pantelides.

Theoretically, if we take a point at infinity, Hauenstein's distance-point method [11] is equivalent to the plane-distance method. But our plane-distance method has the advantage that it is translation invariant. Thus its conditioning does not depend on the distance, as does the most closely related method, the distance-point method of Hauenstein. In addition, in contrast to the approach in [11] for singular critical points, we present a way to move away from a singularity by stepping along a perturbed component to obtain a regular point with hardware precision rather than tracking at singular endpoints with higher precision.

Future research includes loosening the assumptions A_1 and A_2, to address more degenerate cases. Such research includes methods for deflating higher multiplicity components, and also addressing problems with equations which are sums of squares. We will explore the relations with the closest method to ours, that of Hauenstein [11], especially as regards the effect on sparsity of the equation systems for these methods.

Acknowledgements

We thank professor T.Y. Li and Tsung-Lin Lee for their helpful suggestions and the work on Hom4ps2-Maple interface. We also especially thank one of the reviewers for their valuable comments.

6. REFERENCES

[1] S. Basu, R. Pollack and M-F Roy, Algorithms in real algebraic geometry. 2nd edition. Algorithms and Computation in Math., 10. Springer-Verlag, 2006.

[2] D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler, Numerically Solving Polynomial Systems with the Software Package Bertini. In preparation. To be published by SIAM, 2013.

[3] G.M Besana, S. DiRocco, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. Cell decomposition of almost smooth real algebraic surfaces. Num. Algorithms, DOI:10.1007/s11075-012-9646-y, (published online Sept 28, 2012).

[4] B. Buchberger, An Algorithm for Finding the Basis Elements of the Residue Class Ring of a Zero Dimensional Polynomial Ideal. Ph.D. dissertation, University of Innsbruck, 1965.

[5] C. Chen, J.H. Davenport, J.P. May, M.M. Maza, B. Xia and R. Xiao. *Triangular decomposition of semi-algebraic systems* Journal of Symbolic Computation Vol49-0, pp 3-26, 2013.

[6] C. Chen, M. Moreno Maza, B. Xia, L. Yang, Computing Cylindrical Algebraic Decomposition via Triangular Decomposition. Proc. of ISSAC 2009, pages 95-102, ACM Press, New York, 2009.

[7] G. Collins, Quantifier elimination for real closed fields by cylindrical algebraic decomposition. Springer Lec. Notes Comp. Sci. v33. 515-532, 1975.

[8] J.H. Davenport and J. Heintz, Real quantifier elimination is doubly exponential, J. Symbolic Comp., 5:29-35, 1988.

[9] D. Grigorév and N. Vorobjov. Counting connected components of a semialgebraic set in subexponential time. Comput. Complexity, 2(2), 133-186, 1992.

[10] Gunning, R. and Rossi, H. Analytic functions in several complex variables. Eaglewood Cliffs N.J., Prentice Hall, 1965.

[11] J. Hauenstein, Numerically computing real points on algebraic sets. Acta Appl. Math. DOI:10.1007/s10440-012-9782-3, (published online September 27, 2012).

[12] H. Hong. Improvement in CAD-Based Quantifer Elimination. Ph.D. thesis, the Ohio State University, Columbus, Ohio, 1990.

[13] S. Ilie, R.M. Corless, G. Reid, Numerical solutions of index-1 differential algebraic equations can be computed in polynomial time, Numerical Algorithms, 41(2), 161–171, 2006.

[14] T.Y. Li and Tsung-Lin Lee. HOMotopy method for solving Polynomial Systems, software at

`http://www.math.msu.edu/~li/Software.htm`

[15] J.B. Lasserre, M. Laurent and P. Rostalski. A prolongation-projection algorithm for computing the finite real variety of an ideal. Theoret. Comput. Sci., 410(27-29), 2685-2700, 2009.

[16] A. Leykin, J. Verschelde, and A. Zhao. Higher-order deflation for polynomial systems with isolated singular solutions. In Algorithms in algebraic geometry, IMA Volume 146, pages 79-97. Springer, New York, 2008.

[17] Y. Lu, Finding all real solutions of polynomial systems, Ph.D Thesis, University of Notre Dame, 2006. Results of this thesis appear in: (with D.J. Bates, A.J. Sommese, and C.W. Wampler), Finding all real points of a complex curve, Contemporary Mathematics 448 (2007), 183–205.

[18] C. Pantelides. The Consistent Initialization of Differential-Algebraic Systems, SIAM J. Sci. and Stat. Comput. Volume 9, Issue 2, pp. 213-231, 1988.

[19] F. Rouillier, M.-F. Roy, and M. Safey El Din. Finding at least one point in each connected component of a real algebraic set defined by a single equation. J. Complexity, 16 (4), 716-750, 2000.

[20] A. Sard, The measure of the critical values of differentiable maps, Bulletin of the American Mathematical Society 48 (12): 883-890, 1942.

[21] A. Seidenberg, A new decision method for elementary algebra, Ann. of Math. 60, 365-374, 1954.

[22] A.J. Sommese and C.W. Wampler. *The Numerical solution of systems of polynomials arising in engineering and science*. World Scientific Press, 2005.

[23] A. Tarski, A decision method for elementary algebra and geometry, Fund. Math. 17, 210-39, 1931.

[24] J. Visconti. *Numerical Solution of Differential Algebraic Equations, Global Error Estimation and Symbolic Index Reduction.* Ph.D. Thesis. Laboratoire de Modélisation et Calcul. Grenoble. 1999.

[25] W. Wu, G. Reid and S. Ilie, *Implicit Riquier Bases for PDAE and their semi-discretizations.* Journal of Symbolic Computation, Volume 44 Issue 7, Pages 923-941, 2009.

[26] X. Wu and L. Zhi. Computing the multiplicity structure from geometric involutive form, Journal of Symbolic Computation, 47(3): 227-238, 2012.