A note on Geometric Involutive Bases for Positive Dimensional Polynomial Ideals and SDP methods

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ABSTRACT

This paper is motivated by [1] which gives a new symbolic-numeric approach for computing the real radical of zero dimensional polynomial systems using geometric involution and semi-definite programming (SDP) techniques. We explore the interplay between geometric involutive bases and the new SDP methods in the positive dimensional case. An important work on this topic is [5].

Categories and Subject Descriptors

I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms—*Algebraic algorithms*

General Terms

Algorithms

Keywords

numeric, involutive bases, SDP, real radical

1. INTRODUCTION

We consider (finite) systems of ℓ polynomials $P \subset \mathbb{R}[x]$ of degree d in the variables $x = (x_1, ..., x_n)$. Its solution set or variety is $V_{\mathbb{R}}(p_1, ..., p_\ell) = \{x \in \mathbb{R}^n : p_j(x) = 0,\ 1 \leq j \leq \ell\}$. The ideal generated by $P = \{p_1, ..., p_\ell\} \subset \mathbb{R}[x]$ is: $\langle P \rangle_{\mathbb{R}} = \langle p_1, ..., p_\ell \rangle_{\mathbb{R}} = \{f_1 p_1 + ... + f_\ell p_\ell : f_j \in \mathbb{R}[x]\}$. Its associated radical ideal over $\mathbb{R}$ is $\sqrt[\mathbb{R}]{\langle P \rangle} = \{f \in \mathbb{R}[x] : f^{2m} + \Sigma_{j=1}^{s} q_j^2 \in \langle P \rangle \text{ for some } q_j \in \mathbb{R}[x], m \in \mathbb{N}\backslash\{0\}\}$.
Monomials are denoted by $x^\alpha := x_1^{\alpha_1} ... x_n^{\alpha_n}$ where $\alpha \in \mathbb{N}$ and the degree of x^α is $|\alpha| = \alpha_1 + ... + \alpha_n$. Then P can be written: $P = \{\sum_{|\alpha| \leq d} a_{k,\alpha}\, x^\alpha : k = 1...\ell\}$.

Definition 1.1. *Denote the coefficient matrix of P by $C(P) = (a_{k,\alpha})$. Let $\boldsymbol{x}^{(\leq d)}$ be the column vector of monomials x^α with $0 \leq \alpha \leq d$ sorted by total degree. We suppose that the columns of $C(P)$ are sorted in the same order. Then $P = C(P)\boldsymbol{x}^{(\leq d)}$ where $C(P) \in \mathbb{R}^{\ell \times N(n,d)}$ and $N(n,d)$ is the number of monomials in $\boldsymbol{x}^{(\leq d)}$. Polynomials can be equivalently represented by the row vectors of $C(P)$, that is as vectors in $J^d := \mathbb{R}^{N(n,d)}$.*

Definition 1.2. *Let P be (as usual) a finite subset of $\mathbb{R}[x]$ of degree d. The k-th prolongation of system P is $\widehat{\boldsymbol{D}}^k(P) = \{x^\alpha p : 0 \leq \deg(x^\alpha p) \leq d + k, \alpha \in \mathbb{N}^n, p \in P\}$.*

Definition 1.3. *Given a subspace V of J^d and $\ell \leq d$, define $\boldsymbol{\pi}^\ell(V)$ as the vectors of V with the components of degree $\geq d - \ell$ discarded. Given $P \subset \mathbb{R}[x]$ of degree d define $\boldsymbol{\pi}^\ell(P) := \boldsymbol{\pi}^\ell \ker C(P)$. The k-th prolongation of the kernel is $\boldsymbol{D}^k(P) := \ker C(\widehat{\boldsymbol{D}}^k P)$.*

See for example [3] and the published references in [2] for the stable numerical implementations of this paper's operations using SVD methods.

2. GEOMETRIC INVOLUTIVE BASES

In this section we introduce the symbol and Cartan Test necessary for geometric involutive bases.

Definition 2.1. *Suppose $P \subset \mathbb{R}[x]$ of degree d. The symbol matrix $\mathcal{S}(P)$ of P is the submatrix of $C(P)$ corresponding to its degree d monomials. Then the class of a monomial x^α is the least j such that $\alpha_j \neq 0$.*

Suppose that the columns of $\mathcal{S}(P)$ are sorted in descending order by class and that it is reduced to Gauss echelon form. For $k = 1, 2, ..., n$ define the quantities $\beta_d^{(k)}$ as the number of pivots in this reduced matrix of class k. In a generic system of coordinates the symbol is involutive if

$$\sum_{k=1}^{k=n} k\beta_d^{(k)} = \operatorname{rank} \mathcal{S}(\widehat{\mathbf{D}}P) \tag{1}$$

Suppose $Q \subset \mathbb{R}[x]$ has degree d' and a basis for $\ker C(Q)$ is given by the rows of the matrix B. To extract the $\beta_q^{(k)}$ in (1) at projected degree $d \leq d'$ we first numerically project $\ker C(Q)$ onto the subspace J^d by deleting the coordinates in B of degree $> d$ to give a spanning set $\tilde{B}$ for $\boldsymbol{\pi}^{d'-d}Q$. Then delete the columns in $\widetilde{B}$ corresponding to variables of degree $< d$ to obtain a matrix A_d corresponding to the orthogonal complement of the degree d symbol. Let $A_d^{(k)}$ be the submatrix of $\widetilde{B}$ with columns corresponding to variables of class $\leq k$. In generic coordinates for $k = 1 \ldots n$:

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$$\beta_d^{(k)} = \begin{pmatrix} n+d-k-1 \\ d-1 \end{pmatrix} - \left(\text{rank}\, A_d^{(k-1)} - \text{rank}\, A_d^{(k)}\right).$$

Then the SVD can approximate the ranks in this equation for carrying out the Cartan Test (1).

Definition 2.2 (Involutive System). *A system of polynomials $P \subset \mathbb{R}[x]$ is involutive if* $\dim \boldsymbol{\pi D} P = \dim P$ *and the symbol of P is involutive.*

Definition 2.3. *Let $P \in \mathbb{R}[x]$ with $d = \deg P$ and k, ℓ be integers with $k \geq 0$ and $0 \leq \ell \leq k+d$. Then $\boldsymbol{\pi}^\ell \boldsymbol{D}^k P$ is projectively involutive if* $\dim \boldsymbol{\pi}^\ell \boldsymbol{D}^k P = \dim \boldsymbol{\pi}^{\ell+1} \boldsymbol{D}^{k+1} P$ *and the symbol of $\boldsymbol{\pi}^\ell \boldsymbol{D}^k P$ is involutive.*

In [4] we prove that a system is projectively involutive if and only if it is involutive. In the following algorithm we seek the smallest k such that there exists an ℓ with $\boldsymbol{\pi}^\ell \mathbf{D}^k P$ approximately involutive, and generates the same ideal as the input system. We choose the system corresponding to the largest such $\ell \leq k$ if there are several such values for the given k.

Algorithm 2.1 (gif Geometric involutive form).

Input: *$Q \subset \mathbb{R}[x_1, \ldots, x_n]$. A tolerance ϵ.*
Set $k := 0$, $d := \deg(Q)$ and $P := \ker C(Q)$
repeat
  Compute $\boldsymbol{D}^k(P)$
  Initialize set of involutive systems $I := \{\}$
  for** $\ell = 0 \cdots (d+k)$ **do
    Compute $R := \boldsymbol{\pi}^\ell \boldsymbol{D}^k(P)$
    if** R involutive **then** $I := I \cup \{R\}$ **end if
  end do
  Remove systems $\bar{R}$ from I: $\boldsymbol{D}^{d+k-\bar{d}} \bar{R} \not\subseteq \boldsymbol{D}^k(P)$
  $k := k+1$
***until** $I \neq \{\}$*
Output: *Return the polynomial generators of the GIF $\bar{R}$ in I of lowest degree $\bar{d} = \deg \bar{R}$.*

3. MOMENT MATRICES & ALGORITHMS

In this section we outline algorithms for combining geometric involutive form and moment matrix methods.
A moment matrix is a symmetric matrix $M = (M_{\alpha,\beta})$ indexed by $\alpha, \beta \in \mathbb{N}^n$. Here α is the index for rows, β is the index for columns. Without loss $M_{0,0} = 1$. Given $P \subset \mathbb{R}[x_1, ..., x_n]$. Let $d = \deg(P)$ and $M \in \mathbb{R}^{N(n,d)\times N(n,d)}$ be the truncated moment matrix. The linear constraints imposed by P are $M \cdot A^T = 0$; $A = C(\widehat{\mathbf{D}}^d(P))$ where C is the coefficient matrix.

Algorithm 3.1 (GIF – M Method).

Input: $P = \{p_1, ..., p_k\} \subset \mathbb{R}[x_1, \ldots, x_n]$
  $Q_0 := P, \quad j := 0$
  repeat
    $d := \dim \ker \mathtt{gif}(Q_j), \quad Q_{j+1} := \mathtt{gen}(\mathtt{gif}(Q_j))$
    $r := \text{rank}(\mathtt{M}(Q_{j+1})), \quad Q_{j+2} := \mathtt{gen}(\ker \mathtt{M}(Q_{j+1}))$
    $j := j+2$
  until $r = d$
Output: $Q = \{q_1, ..., q_\ell\} \subset \mathbb{R}[x_1, \ldots, x_n]$
Q is in geometric involutive form
$\sqrt[\mathbb{R}]{\langle P \rangle_{\mathbb{R}}} \supseteq \langle Q \rangle_{\mathbb{R}} \supseteq \langle P \rangle_{\mathbb{R}}$.

The algorithm above uses the following subroutines.

Algorithm 3.2 (M - Moment Matrix).

Input: $Q \subset \mathbb{R}[x_1, \ldots, x_n]$. Set $d := \deg(Q)$.
1. Construct the moment matrix to degree $2d$.
2. Prolong Q to degree $2d$ and form $\mathtt{M}(Q)$.
3. Use SDP methods to numerically solve for a generic point that maximizes the rank of the moment matrix subject to the constraints.

Output: Return $\mathtt{M}(Q) \succeq 0$ the moment matrix evaluated at this generic point.

Algorithm 3.3 (GEN).

Input: $\mathtt{gif}(Q)$ or $\ker \mathtt{M}(Q)$
Output: Polynomial generators corresponding to $\mathtt{gif}(Q)$ or $\ker \mathtt{M}(Q)$

Rank-Dim-Involutive Stopping Criterion: A natural termination criterion used in Algorithm 3.1 is that the generators stabilize at some iteration and the system is involutive:

$$\mathtt{gen}(\mathtt{gif}(Q)) = \mathtt{gen}(\ker \mathtt{M}(Q)) \text{ and } Q \text{ involutive} \qquad (2)$$

By [1] $\langle \mathtt{gen}(\ker \mathtt{M}(Q_{j+1})) \rangle$ is a sequence of ideals containing $\sqrt[\mathbb{R}]{\langle P \rangle}$. We get an ascending chain of ideals in a Noetherian ring $\mathbb{R}[x_1, ..., x_n]$. Hence, together with the finiteness of the Cartan-Kuranishi geometric involutive form algorithm, Algorithm 3.1 terminates.

4. CLOSING REMARKS

Significant progress is made [5] where variations of Pommaret involutive bases are used with SDP moment matrices calculations to determine (at least some) members of the real radical in positive dimension. In a delta regular (generic) coordinate systems [5] uses a stopping criterion based on comparing a difference of moment matrix co-ranks with the Cartan critical rank (1).
The degree of the geometric involutive basis in our method can be lower than that given in [5] since Algorithm 3.1 updates the generators with projections. However in the absence of a proof of determination of the real radical the larger moment matrices of [5] can capture new members of the real radical in situations where our method has already terminated.
Additional discussion and examples are given in the long version of our work [2].

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