A Recursive Decision Method for Termination of Linear Programs*

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ABSTRACT

In their CAV 2004 and 2006 papers, Tiwari and Braverman have proved that, for a class of linear programs over the reals, termination is decidable. In this paper, we propose a new algorithm to decide whether a program of the same class terminates or not. In our approach, a program with an assignment matrix having a single Jordan block or having several Jordan blocks with the same eigenvalue is treated as a basic program to which we reduce a program with arbitrary assignment matrices in a recursive process. Furthermore, if a basic program is non-terminating, our method constructs at least one point on which a given basic program does not terminate. In contrast, for a non-terminating basic program, in most cases, the methods of Tiwari and Braverman provide only a so-called N-nonterminating point. Also, different from their methods, we do not need to guess a dominant term from every loop condition in our recursive procedure.

Keywords

Linear Loops, Termination Analysis, Semi-algebraic sets

Categories and Subject Descriptors

D.2.4 [**Software/Program Verification**]: [Reliability, Validation]; F.3.1 [**Specifying and Verifying and Reasoning about Programs**]: [Specification techniques]

General Terms

Reliability, Experimentation, Security, Theory, Verification.

Keywords

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1. INTRODUCTION

It is well known that guaranteeing software systems trustworthy is a grand challenge in theoretical computer science. As one of the building blocks of automated program verification, termination analysis has attracted increasing interest in the recent years. However, the termination problem is undecidable in most cases. Therefore, most well-established work concentrates on the construction of well-founded ranking functions. Especially, Poldelski and Rybalchenko [9] first presented a complete method for the synthesis of linear ranking functions in 2004. But, it has been shown that the existence of ranking functions is just a sufficient (but not necessary) condition for guaranteeing the termination of loops. That is to say, one can construct an example of a loop that terminates but has no ranking function. Because of the reasons mentioned above, people pay attention to explore a decidable class of loops. For example, Tiwari [10] in 2004 showed that the termination of a linear loop program of the following shape is decidable, as follows:

$$P_1 \quad while \;\; (BX > 0) \;\; \{X := AX\}$$

Where $B \in \mathbb{R}^{m\times n}$ is called the condition matrix, and $A \in \mathbb{R}^{n\times n}$ the assignment matrix. This classic work shows us new insight on termination problem for programs. In 2006, Braverman [3] generalized the work of Tiwari and proved that the loops of the above class is also decidable over the integers. Following their work, Xia et al. [12] considered the termination of a more general class of loops with nonlinear constraints and linear updates. They proved that under proper conditions, such loops were decidable over the reals. Since the decision procedure given by Tiwari depends on the computation of Jordan canonical forms, Yang et al. [11] presented a purely symbolic method to compute Jordan canonical forms.

In this paper, we reconsider the same termination problem proposed and analyzed by Tiwari in 2004. And we present a practical algorithm for the termination of program P_1. Our main contributions in this paper are as follows. First, we give a simple method to decide the termination of a special class of linear loops, whose assignment matrices consist only of one Jordan block with positive real eigenvalue, i.e., $A = J(\lambda), \lambda > 0$. Furthermore, for these special loops, we construct a subset of the set of nonterminating points, which enables us to analyze the termination of this kind of loops only by determining whether the subset is empty or not. This result can also be generalized to determine the termination of another class of linear programs, whose assignment matrices consist only of finitely many Jordan blocks with the same eigenvalue. Second, for the general case, i.e., $A = diag(J_1(\lambda_1), ..., J_s(\lambda_s))$, $\lambda_i > 0$, for $i = 1, ..., s$, a recursive decision process is developed to analyze the termination of them. It has been pointed out that although a linear program may not be presented in this form, termination problem can always be reduced to this form by [10].

The rest of the paper is organized as follows. In Section 2, we recall some important results presented in [10]. In Section 3, according to the number of Jordan blocks in assignment matrices, the termination of linear loops are divided into two cases: the first case, i.e., $A = J$ where J is a Jordan block with positive real eigenvalue; the second case, i.e., $A = diag(J_1, ..., J_s)$ where J_i's are all Jordan blocks with positive real eigenvalue. Decision methods are established to analyze the termination of the two classes of loops. Moreover, an example is given in Section 4 to illustrate our methods. Finally, we conclude the paper in Section 5.

2. PREVIOUS RESULTS

In [10], Tiwari establishes the decidability of the termination problem for linear loops of the form P_1. Generally speaking, we say that Program P_1 is nonterminating over the reals, if there is a point $X \in \mathbb{R}^n$, such that $BA^nX > 0$ holds for all $n \geq 0$. Otherwise, if such a point does not exist, then Program P_1 terminates over the reals. Especially, we say Program P_1 is nonterminating over a set $\Omega \subset \mathbb{R}^n$, if there is a point $X \in \Omega$, such that $X(n) = A^nX \in \Omega$ and $BX(n) > 0$ holds for all $n \geq 0$. For convenience, let $Cond(n, X)$ be the expression after n iterations of loop condition in Program P_1, i.e., $Cond(n, X) = BA^nX$. Define $NT = \{X \in R^n : BX > 0, BAX > 0, BA^2X > 0, \ldots, BA^iX > 0, \ldots\}$ to be the set of all points on which Program P_1 does not terminate. Each point in NT is called nonterminating point. At the same time, we denote by NT^e the set of X's for which $A^NX \in NT$ for some N. For convenience, we call every point in NT^e N-nonterminating point. Now let us introduce two important results from [10], which can reduce the termination problem of Program P_1 to that of a simper class of programs.

THEOREM 1. *([10]) Let P be an invertible matrix. Then the program*

$$\boldsymbol{P_1}: \quad while\ (BX > 0)\ \{X := AX\}$$

is terminating if and only if the program

$$\boldsymbol{P_2}: \quad while\ (BPY > 0)\ \{Y := P^{-1}APY\}$$

is terminating.

By Theorem 1, the termination program of Program $\boldsymbol{P}_1$ can be reduced to determining the termination of Program $\boldsymbol{P}_2$. Let $P^{-1}AP = diag(J_1, \ldots, J_k)$, where J_i is the i-th Jordan block in the Jordan canonical form of the assignment matrix A. Let λ_i be the eigenvalue corresponding to the i-th Jordan block J_i. Partition the variables in Y into k segments, say, $Y_1, ..., Y_k$, and Program $\boldsymbol{P}_2$ can be rewritten as the form:

$$\begin{aligned} \boldsymbol{P_3}: \quad & while(B_1Y_1 + \cdots + B_kY_k > 0) \\ & \{Y_1 := J_1Y_1; \ldots; Y_k := J_kY_k\}. \end{aligned}$$

Where B_i's are obtained by partitioning the matrix BP. Let $\Lambda = \{1, 2, \ldots, k\}$ be the set of indices. Define the set $\Lambda_+ = \{i \in \Lambda : \lambda_i > 0\}$. The following theorem presented in [10] shows that we can ignore the state space corresponding to negative and complex eigenvalues.

THEOREM 2. *([10]) Let P be an invertible matrix. The program*

$$\boldsymbol{P_3}: \quad while\ (\sum_{j\in\Lambda} B_jY_j > 0)\ \{Y_j := J_jY_j;\ j \in \Lambda\}$$

is terminating if and only if the program

$$\boldsymbol{P_4}: \quad while\ (\sum_{j\in\Lambda_+} B_jY_j > 0)\ \{Y_j := J_jY_j;\ j \in \Lambda_+\}$$

is terminating.

In terms of Theorem 2, we know that if the Program $\boldsymbol{P}_4$ does not terminate on input $Y_j := \mathbf{c_j}$ $(j \in \Lambda_+)$,then the Program $\boldsymbol{P}_3$ does not terminate on input $Y_j := \mathbf{c_j}(\ j \in \Lambda_+)$ and $Y_j := 0(\ j \notin \Lambda_+)$. Moreover, Theorem 2 tells us that we can reduce the termination of Program $\boldsymbol{P}_1$ to the termination of Program $\boldsymbol{P}_4$. It also shows that we just need to consider the eigenspaces corresponding to positive eigenvalues of A. Thus, in the following, we just need to consider the termination of the following program:

$$\mathfrak{J}_1: \quad while\ (BX > 0)\ \{X := AX\}$$

where $A = diag(J_1, ..., J_m)$ is a diagonal matrix and J_i is a Jordan block with positive real eigenvalue λ_i, for $i = 1, ..., m$. If $m = 1$, then we see that the assignment matrix A of $\mathfrak{J}_1$ consists only of one Jordan block, i.e., $A = J$. In the following subsection 3.1, we present two different methods for deciding the termination of this special form of Program $\mathfrak{J}_1$, and the termination for Program $\mathfrak{J}_1$ is analyzed in subsection 3.2.

3. TERMINATION DECISION OF PROGRAM $\mathfrak{J}_1$

In the section, we first consider two special classes of loops and establish decision methods for the termination of them. Then, based on these methods, a recursive decision method is presented to determine the termination of a more general class of loops.

3.1 The Special Case: $A = J$

In the subsection, we consider the termination for the special form $\mathfrak{J}_0$ of $\mathfrak{J}_1$, as follows:

$$\mathfrak{J}_0: \quad while\ (BX > 0)\ \{X := AX\}$$

Where $B \in \mathbb{R}^{s\times r}, X \in \mathbb{R}^{r\times 1}, A = J_{r\times r} \in \mathbb{R}^{r\times r}$, and J is a Jordan block with positive real eigenvalue λ, in the following form:

$$A_{r\times r} = J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \quad (\lambda > 0) \qquad (1)$$

By knowledge in linear algebra, we know that the n-th power of the matrix J can be explicitly written as

$$J^n = A^n_{r\times r} = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & C_n^2\lambda^{n-2} & \cdots & C_n^{r-1}\lambda^{n-(r-1)} \\ 0 & \lambda^n & n\lambda^{n-1} & \cdots & C_n^{r-2}\lambda^{n-(r-2)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \cdots & 0 & \lambda^n \end{pmatrix}$$

Where $\lambda > 0$, r is the dimension of the Jordan block J, and $C_n^k = 0$ for $n < k$.

Given two integers $a, b (a < b)$, define $\overline{[a, b]} = \{a, a+1, ..., b-1, b\}$. Let $\mathbb{Z}_{\geq 0}$ be a set of nonnegative integers. Let $b^i = (b_{i1}, ..., b_{ir})$ be the i-th row of B, and let $\mathrm{X} = (x_1, x_2, ..., x_r)^T$.

THEOREM 3. *With the above notion. Program $\mathfrak{J}_0$ is terminating if and only if there exist $i, j \in \overline{[1, s]}$, such that $b_{ik_i} \cdot b_{jk_j} < 0$. Where b_{ik_i}, b_{jk_j} are the leftmost nonzero entries of b^i and b^j, respectively.*

PROOF. Assume that there exist two integers $k_i, k_j \in \overline{[1, r]}$, for $i, j \in \overline{[1, s]}$, such that $b_{ik_i} \cdot b_{jk_j} < 0$, where b_{ik_i}, b_{jk_j} are the leftmost nonzero elements of b^i, b^j respectively. Therefore, after n iterations, the i-th and the j-th loop conditions can be written as $b^i A^n X$ and $b^j A^n X$, respectively. Since both of the above expressions can be divisible by $\lambda^{n-(r-1)}$, we define

$$p_i(\mathrm{X}, n) = \frac{b^i A^n X}{\lambda^{n-(r-1)}}, \qquad p_j(\mathrm{X}, n) = \frac{b^j A^n X}{\lambda^{n-(r-1)}}.$$

It is clear that p_i, p_j can be regarded as polynomials in n, i.e. $p_i = p'_i(n)$, $p_j = p'_j(n)$, where $p'_i, p'_j \in \mathbb{R}[\mathrm{X}][n]$. By the construction of $p'_i(n)$ and $p'_j(n)$, it is easy to observe that the leading coefficients of $p'_i(n)$ and $p'_j(n)$ w.r.t n are $\mu_i^{(r)} \cdot b_{ik_i} x_r$ and $\nu_j^{(r)} \cdot b_{jk_j} x_r$ with $\mu_i^{(r)} > 0, \nu_j^{(r)} > 0$, respectively. Hence, when $n \to \infty$, the sign of $p_i(n)$ and $p_j(n)$ must depend totally on that of the two terms $b_{ik_i} x_r$ and $b_{jk_j} x_r$. Next, for the value of x_r, consider two cases:

Case 1. if $x_r \neq 0$, then Program $\mathfrak{J}_0$ must terminate on the point $X = (x_1, x_2, \ldots, x_{r-1}, x_r \neq 0)^\tau$ in that the product $(\mu_i^{(r)} \cdot b_{ik_i} x_r) \cdot (\nu_j^{(r)} \cdot b_{jk_j} x_r)$ of the two leading coefficients is negative. Thus, when $n \to \infty$, the signs of p_i, p_j are different, which implies that Program $\mathfrak{J}_0$ must terminate on the point $X = (x_1, x_2, \ldots, x_{r-1}, x_r \neq 0)^\tau$.

Case 2. if $x_r = 0$, then $\mu_i^{(r-1)} \cdot b_{ik_i} x_{r-1}$ and $\nu_j^{(r-1)} \cdot b_{jk_j} x_{r-1}$ are exactly the leading coefficients of $p_i(n), p_j(n)$ with $\mu_i^{(r-1)} > 0, \nu_j^{(r-1)} > 0$. Likewise, when $n \to \infty$, the sign of p_i and p_j must be dependent totally on that of the two terms $b_{ik_i} x_{r-1}$ and $b_{jk_j} x_{r-1}$ respectively. Analogically, we can discuss the value of x_{r-1} as above. Repeating this process, we can conclude that when $b_{ik_i} \cdot b_{jk_j} < 0$, Program $\mathfrak{J}_0$ must be terminating.

For the converse, assume that for any $i, j \in \overline{[1, s]}$, we have $b_{ik_i} \cdot b_{jk_j} > 0$. That is, all b_{jk_j}'s have the same sign. Since the term $\mu_i \cdot b_{ik_i} x_r$ with $\mu_i > 0$, is exactly the leading coefficients of $p_i(n)$, for large n, the sign of p_i will be dependent only on that of the term $b_{ik_i} x_r$. Hence, let $x_k = 0$ for all $k \in \overline{[1, r-1]}$, and let $x_r = x_r^0$ satisfying $b_{ik_i} x_r^0 > 0$, for all $i = 1, ..., s$. An N-nonterminating point X^0 is obtained, say $X^0 = (0, \ldots, 0, x_r^0)^T$. That is to say, for large n, say $n = N$, we can obtain a nonterminating point $X = A^N X^0$. Obviously, this contradicts the condition of the theorem. Thus, this assumption does not hold. we complete the proof of the theorem. □

By Theorem 3, a simple corollary can be derived as follows.

COROLLARY 1. *Let $\Omega_k = \{X \in \mathbb{R}^r : x_{k+1} = \cdots = x_r = 0\}$. Suppose that $b_{i1} \neq 0$ for all $i = 1, ..., s$. Then, Program $\mathfrak{J}_0$ is terminating over the set Ω_k if and only if there exist $i, j \in \overline{[1, s]}$, such that $b_{i1} \cdot b_{j1} < 0$.*

Example 1. Consider the linear program from [10]:

$$Q_1: \ \textit{while}\ (x > 0 \wedge y > 0)\ \{x := x - y;\ y := y\}$$

Define A' to be the assignment matrix of Q_1, i.e., $A' = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. So, the real Jordan canonical form of A' is $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the corresponding transformation matrix is $P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. In terms of Theorem 1, the termination analysis of Q_1 can be converted to that of Q_2, as follows.

$$Q_2: \ \textit{while}\ (-x > 0 \wedge y > 0)\ \{x := x + y;\ y := y\}$$

To decide the termination of Q_1, we just need to determine that of Q_2. It is obvious to see that the assignment matrix of Program Q_2 consists only of one Jordan block. That is to say, Q_2 belongs to this type of loops described as $\mathfrak{J}_0$. By Theorem 3, it follows that Program Q_2 is terminating since $-1 \cdot 1 < 0$. Thus, we conclude that Program Q_1 is terminating by Theorem 1.

Remark 1. By Theorem 3, one can easily analyze the termination of Program $\mathfrak{J}_0$. However, the method can not give us at least one nonterminating point, if they indeed do not terminate. In Subsection 3.2, a subset of the set of nonterminating points of Program $\mathfrak{J}_0$ is characterized by semi-algebraic systems. This enables us to get the desired nonterminating points by solving these systems, if the subset is nonempty.

The following proposition will play a critical role in the following text. We now state it here as follows.

PROPOSITION 1. *Given a polynomial $h(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0, h(x) \in \mathbb{R}[x]$. Let $\xi(x,\ b) = h(x+b) (b \geq 0)$. Then, when $b \to +\infty$, there exists a real number $b^* \geq 0$, such that for any $b \geq b^*$, the number of variation of coefficients of $\xi(x,\ b)$ is zero.*

PROOF. The proof is simple. Set

$$\begin{aligned} \xi(x, b) &= h(x + b) \\ &= a_n (x+b)^n + a_{n-1} (x+b)^{n-1} + \cdots + a_1 (x+b) + a_0 \\ &= \beta_n(b) x^n + \beta_{n-1} x^{n-1} + \cdots + \beta_1(b) x + \beta_0(b) \end{aligned}$$

Next, we first claim that $lcoeff(\beta_j(b), b) = c_n^{n-j} a_n$, for $j = 0, 1, ..., n$. Where, $lcoeff(\diamond, \circ)$ is a function, which returns the leading coefficient of $\diamond$ w.r.t the indeterminate $\circ$. Collecting terms from $\xi(x, b)$ according to the exponents of x, we have

$$\begin{aligned} \xi(x,b) = {} & a_n x^n + (a_n c_n^1 b + a_{n-1}) x^{n-1} + (a_n c_n^2 b^2 + \\ & a_{n-1} c_{n-1}^1 b + a_{n-2}) x^{n-2} + \cdots + (a_n b^n + a_{n-1} b^{n-1} \\ & + \cdots + a_1 b + a_0) \\ = {} & \sum_{j=0}^{n} \sum_{i=j}^{n} (c_i^{i-j} a_i b^{i-j}) x^j \end{aligned}$$

Clearly, for all $j = 0, 1, ..., n$, we have $lcoeff(\beta_j(b), b) = C_n^{n-j} a_n$. Next, there are two cases to be considered, $a_n > 0, a_n < 0$. We just consider the first case when $a_n > 0$, and the analysis of the second case is similar to that of the first case.

(I) When $a_n > 0$. since a_n lies in the leading coefficient of $\beta_j(b)$, we have $\beta_j(b) \to +\infty$, as $b \to +\infty$. Thus, for any a given $G_j > 0$, there must exist $B_j > 0$, such that when $b > B_j$, we have $\beta_j(b) > G_j > 0$. Set $b^* = max\{B_0, B_1, ..., B_{n-1}\}+1$, when $b \geq B^*$, we have $\beta_j(b) > 0$, for each $j \in \overline{[0, n-1]}$. It immediately follows that when $b \geq B^*$, the number of variation of coefficients of $\xi(x,\ b)$ becomes zero. This completes this proposition. □

According to Proposition 1, we have

$\mathfrak{a}$) if $a_n > 0$, then there must exist $b \in \mathbb{Z}_{\geq 0}$ satisfying the semi-algebraic system $\mathfrak{S}_+ = \{\beta_n(b) > 0, \beta_{n-1}(b) > 0, \ldots, \beta_1(b) > 0, \beta_0(b) > 0, b \geq 0\}$.

$\mathfrak{b}$) if $a_n < 0$, then there must exist $b \in \mathbb{Z}_{\geq 0}$ satisfying the semi-algebraic system $\mathfrak{S}_- = \{\beta_n(b) < 0, \beta_{n-1}(b) < 0, \ldots, \beta_1(b) < 0, \beta_0(b) < 0, b \geq 0\}$.

It is easy to see that any nonzero solution of $\mathfrak{S}_+ (or\ \mathfrak{S}_-)$ can be regarded as a bound for positive roots of $h(x)$ by Descartes' rule of signs.

In general, finding the set NT is difficult since the behavior of a program is very complicated. However, for Programs $\mathfrak{J}_0$, a method is established to construct a subset NT' of NT. First, let us consider the below program $\mathfrak{J}_0'$ with only one loop condition.

$$\mathfrak{J}_0' : \quad while \ \ (C^T X > 0) \ \ \{X := AX\}.$$

Where $C \in \mathbb{R}^{1\times r}$, X $\in \mathbb{R}^r$, and the assignment matrix A of Program $\mathfrak{J}_0'$ is the same with that of $\mathfrak{J}_0$. Therefore, Program $\mathfrak{J}_0'$ can be regarded as a special case of Program $\mathfrak{J}_0$. Denote by $cond(n, X)$ the expression of its loop condition after n iterations. Namely, $cond(n, X) = C^T A^n X = \lambda^n f(n, X)$, where $f(n, X) = \sum_{i=0}^{r-1} a_i(X)n^i$. For example, consider the following linear loop, $while(3x_1 + 4x_2 > 0) \ \ \{X := AX\}$, where $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\lambda > 0$. Let $C = (3, 4)^T$. The loop condition of this loop after n-th iteration can be written as the form,

$$\begin{aligned} Cond(n, X) = C^T A^n X &= (3,4) \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} (x_1, x_2)^T \\ &= \lambda^n (3\lambda^{-1} x_2 n + 3x_1 + 4x_2) = \lambda^n f(n, X). \end{aligned}$$

Where $f(n, X) = a_1(X)n + a_0(X) = (3\lambda^{-1}x_2)n + (3x_1 + 4x_2)$.

We say Program $\mathfrak{J}_0'$ is nonterminating, if there exists at least one point X^*, such that for any $n \in \mathbb{Z}_{\geq 0}$, we have $Cond(n, X^*) > 0$. Since $\lambda > 0$, $Cond(n, X^*) > 0$ is equivalent to $f(n, X^*) > 0$. Therefore, to determine if Program $\mathfrak{J}_0'$ is nonterminating is to check if there exists at least one point X^*, such that for any $n \in \mathbb{Z}_{\geq 0}$, $f(n, X^*) > 0$ holds.

Before we state the following theorem, some notions need to be given first.

Theorem 4. *With the above notion. Program $\mathfrak{J}_0'$ is nonterminating if and only if there exists $k \in \overline{[0, r-1]}$, such that the following semi-algebraic set*

$$\begin{aligned} S_k = \{X \in \mathbb{R}^r : \ & a_v(X) = 0 \ for \ all \ v > k, \\ & a_v(X) > 0 \ for \ all \ v \leq k\} \end{aligned}$$

has solutions.

Proof. Suppose that there exists k, such that S_k has at least one solution, say $X^* \in S_k$. Then for all $n \in \mathbb{Z}_{\geq 0}$, the sign of $f(n, X^*)$ having degree k w.r.t n remains positive. Hence, Program $\mathfrak{J}_0'$ does not terminate on X^*. For the converse, assume that Program $\mathfrak{J}_0'$ does not terminate on $X^* \in NT$. That is, for any $n \in \mathbb{Z}_{\geq 0}$, $f(n, X^*) > 0$. Next, we show that there exists a certain $k \in \overline{[0, r-1]}$ such that S_k is nonempty. Without loss of generality, suppose that the degree of $f(n, X^*)$ is k. So, we have $a_v(X^*) = 0$ for all $v > k$ and $a_k(X^*) > 0$. Since $C^T A^{n+b} X^* = \lambda^{n+b} f(n+b, X^*)$, by proposition 1, there exists $b = b^* \in \mathbb{Z}_{\geq 0}$, such that the coefficients of $f(n+b^*, X^*)$ are all positive. Hence, for all $n \in \mathbb{Z}_{\geq 0}$, we have $f(n + b^*, X^*) > 0$. Since $\lambda^{n+b^*} f(n + b^*, X^*) = C^T A^{n+b^*} X^* = C^T A^n (A^{b^*} X^*) = \lambda^n f(n, A^{b^*} X^*)$, $(\lambda > 0)$,we know that $f(n, A^{b^*} X^*) = \lambda^{b^*} f(n + b^*, X^*)$. By the analysis of the sign of the coefficients of $f(n+b^*, X^*)$ stated above, it follows that $Y^* = A^{b^*} X^*$ satisfies all of the inequalities in S_k. The remaining task is to claim that Y^* also satisfies the equations in S_k. To prove this, substituting $X = Y^*$ into $C^T A^n X$, we have $\lambda^n f(n, Y^*) = \lambda^n f(n, A^{b^*} X^*) = C^T A^{n+b^*} X^* = \lambda^{n+b^*} f(n + b^*, X^*)$, $(\lambda > 0)$. Since $\lambda^n f(n, X) = \lambda^n \sum_{i=0}^{r-1} a_i(X)n^i$, we can get that $\lambda^n f(n, Y^*) = \lambda^n \sum_{i=0}^{r-1} a_i(Y^*)n^i = \lambda^{n+b^*} f(n+b^*, X^*) = \lambda^{n+b^*} \sum_{i=0}^{r-1} \beta_i(X^*)n^i$. It is not difficult to see that $\beta_i(X^*) = c_i a_i(X^*) + \cdots + c_{r-1} a_{r-1}(X^*)$, for $i = 0, ..., r-1$. Where $c_0, ..., c_{r-1} \in \mathbb{R}$. Therefore, since $a_i(X^*) = 0$ for all $i > k$, we have $\beta_{k+1}(X^*) = \cdots = \beta_{r-1}(X^*) = 0$. It immediately follows that $a_{k+1}(Y^*) = \cdots = a_{r-1}(Y^*) = 0$. Thus, $Y^* \in S_k$. This completes the proof of the theorem. □

Remark 2. By Theorem 4, it is not difficult to see that any solution of S_k, for $k \in \overline{[0, r-1]}$, is a non-terminating point on which Program $\mathfrak{J}_0'$ does not terminate.

Analogous analysis can be naturally generalized to Program $\mathfrak{J}_0$. Since the number of loop conditions is greater than 1, say $s(s > 1)$, after substituting $X(n) = A^n X$ into s loop conditions, we get s polynomials $f_i(n, X)$, for $i = 1, ..., s$. Similarly, construct sr semi-algebraic sets, say $S^1_{k_1}, S^2_{k_2}, ..., S^s_{k_s}$, for $k_i \in L = \overline{[0, ..., r-1]}$.

Corollary 2. *With the above notation. Program $\mathfrak{J}_0$ is nonterminating iff there exists an s-tuple $(k_1, k_2, ..., k_s) \in L^s$, such that*

$$S^1_{k_1} \bigcap S^2_{k_2} \bigcap \cdots \bigcap S^s_{k_s} \neq \emptyset.$$

Remark 3. It is easy to see that any solution of $S^1_{k_1} \bigcap S^2_{k_2} \bigcap \cdots \bigcap S^s_{k_s}$ is a non-terminating point on which Program $\mathfrak{J}_0$ does not terminate.

Example 2. Consider the following linear program.

$$\begin{aligned} Q_3 : \ while\ (3x_1 + 4x_2 > 0 \wedge -x_1 + x_2 > 0) \\ \{x_1 := x_1 + x_2;\ x_2 := x_2\} \end{aligned}$$

Since the assignment matrix is a Jordan block with a positive real eigenvalue 1, the termination of Q_3 can be determined by Corollary 2. According to Corollary 2, we construct 4 semi-algebraic systems: $S^1_0 = \{(x_1, x_2) \in \mathbb{R}^2 : \ 3x_2 = 0, 3x_1 + 4x_2 > 0\}$, $S^1_1 = \{(x_1, x_2) \in \mathbb{R}^2 : \ 3x_2 > 0, 3x_1 + 4x_2 > 0\}$, $S^2_0 = \{(x_1, x_2) \in \mathbb{R}^2 : \ -x_2 = 0, -x_1 + x_2 > 0\}$ $S^2_1 = \{(x_1, x_2) \in \mathbb{R}^2 : \ -x_2 > 0, -x_1 + x_2 > 0\}$. Let $L = [0, 1]$. It is easy to see that for any 2-tuple $(k_1, k_2) \in L^2$, $S^1_{k_1} \bigcap S^2_{k_2} = \emptyset$. This implies that Q_3 is terminating. The same conclusion can also be drawn by Theorem 3.

3.2 The General Case: $A = diag(J_1, J_2, ..., J_m)$

In this subsection, we analyze the termination of the below Program $\mathfrak{J}_1$, which has the following general form.

$$\mathfrak{J}_1: \quad while\ (BX > 0)\ \{X := AX\}.$$

Where $A = diag(J_1, J_2, ..., J_m)$ with each Jordan block J_i having the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix} \quad (\lambda_i > 0) \qquad (2)$$

Let r_i be the dimension of J_i. Thus, the dimension of A is $r = \sum_{i=1}^{m} r_i$. And B is an $s \times r$ real matrix and can be written as $B = (b^1, ..., b^s)^T$, and $b^i = (b_1^i, ..., b_m^i)$, for $i \in \overline{[1, s]}$, Then the variable vector X may be partitioned into m subvectors, according to the dimension of each Jordan block J_i, i.e. $X = (\mathbf{x}_1, ..., \mathbf{x}_m)^T$.

We say Program $\mathfrak{J}_1$ is nonterminating, if there exists at least one point X^*, such that for any $n \in \mathbb{Z}_{\geq 0}$, we have $Cond(n, X^*) = BA^nX^* > 0$.

First, we consider a special case of Program $\mathfrak{J}_1$: all eigenvalues of A are equal. To avoid confusion of symbol, we denote by $\mathfrak{J}_1'$ this special program. The termination analysis of $\mathfrak{J}_1'$ is similar to that of $\mathfrak{J}_0$. For Program $\mathfrak{J}_1'$, Since $A^n = diag(J_1^n, ..., J_m^n)$, substituting $X = A^nX$ into s loop conditions, we get new expressions, $BA^nX = (\lambda^n p_1(n, X), ..., \lambda^n p_s(n, X))^T$. And $p_j(n, X)$'s can be regarded as polynomials in n. For $j = 1, ..., s$, let the degree of $p_j(n, X)$ w.r.t n is d_j. Set $L_j = [0, 1, ..., d_j]$. So, $p_j(n, X) = \sum_{i=0}^{d_j} a_{ij}(X)n^i$. Let

$$\begin{aligned} s_{k_j}^j = \{ & X \in \mathbb{R}^r : a_{i,j}(X) = 0\ for\ all\ i > k_j, \\ & a_{i,j}(X) > 0\ for\ all\ i \leq k_j \} \end{aligned}$$

Then, construct $\sum_{j=1}^{s}(1+d_j)$ semi-algebraic sets $S_{k_1}^1, ..., S_{k_s}^s$, $k_j \in L_j$. As stated in Theorem 4 and corollary 2, a similar result is established as follows.

Corollary 3. *With the above notion. Program $\mathfrak{J}_1'$ is nonterminating if and only if there exists an s-tuple $(k_1, k_2, ..., k_s) \in \prod_{i=1}^{s} L_i$, such that $S_{k_1}^1 \bigcap S_{k_2}^2 \bigcap \cdots \bigcap S_{k_s}^s \neq \emptyset$.*

Proof. The proof is similar to what we did in Theorem 4 and we will still use the strategy (and notation) of proof of Theorem 4. One direction of the proof is obvious since $S_{k_1}^1 \bigcap S_{k_2}^2 \bigcap \cdots \bigcap S_{k_s}^s \neq \emptyset$ implies that $\mathfrak{J}_1'$ is nonterminating. For the converse, assume that there is a point $X^* = (\mathbf{x}_1, \mathbf{x}_2, .., \mathbf{x}_m)$, on which $\mathfrak{J}_1'$ does not terminate. Without loss of generality, let us consider the jth loop condition. We claim that there is a $k_j \in L_j$ such that $S_{k_j}^j$ is nonempty. Substituting $X(n) = A^nX$ into the jth loop condition, we have $b^jA^nX = b_1^jJ_1^n\mathbf{x}_1 + \cdots + b_m^jJ_m^n\mathbf{x}_m = \lambda^n p_{j1}(n, \mathbf{x}_1) + \cdots + \lambda^n p_{jm}(n, \mathbf{x}_m) = \lambda^n p_j(n, X) = \lambda^n \sum_{i=0}^{d_j} a_{ij}(X)n^i$. Substitute $X = X^*$ into $p_j(n, X)$. Denote by $k_j \in L_j$ the degree of $p_j(n, X^*)$. It follows that $a_{i,j}(X^*) = 0$ for all $i > k_j$. Since Program $\mathfrak{J}_1'$ does not terminate on X^*, the leading coefficient $a_{k_j}(X^*)$ of $p_j(n, X^*)$ must be a positive number. By Proposition 1, we know that there is $b_j^* \in \mathbb{Z}_{\geq 0}$ such that the number of variation of the coefficients of $p_j(n + b_j^*, X^*)$ is zero. Namely, the coefficients of $p_j(n + b_j^*, X^*)$ are all positive. Note that $b^jA^{n+b_j^*}X^* = \lambda^{n+b_j^*}p_j(n + b_j^*, X^*) = \lambda^n p_j(n, A^{b_j^*}X^*)$. So, there exists $Y_j^* = A^{b_j^*}X^*$ such that $a_{i,j}(Y_j^*) > 0\ for\ all\ i \leq k_j$. Next, we show that Y_j^* satisfies the equations $a_{i,j}(X) = 0$ for all $i > k_j$. To prove this, substituting $X = Y_j^*$ into b^jA^nX, we get

$$\begin{aligned} \lambda^n p_j(n, A^{b_j^*}X^*) = & b^jA^{n+b_j^*}X^* = \lambda^{n+b_j^*}p_j(n + b_j^*, X^*) \\ & (\lambda > 0) \end{aligned} \qquad (3)$$

Since $\lambda^n p_j(n, X) = \lambda^n \sum_{i=0}^{d_j} a_{ij}(X)n^i$, we have

$$\lambda^{n+b_j^*}p_j(n + b_j^*, X^*) = \lambda^{n+b_j^*} \sum_{i=0}^{d_j} \beta_{ij}(X^*)n^i.$$

It is not difficult to see that $\beta_{ij}(X^*) = c_{ij}a_{ij}(X^*) + \cdots + c_{d_j,j}a_{d_j,j}(X^*)$, for $i = 0, ..., d_j$. Where $c_{ij}, ..., c_{d_j,j} \in \mathbb{R}$. So, if $a_{i(i>k_j),j}(X^*) = 0$, we have $\beta_{i,j}(X^*) = 0$, for $i = k_j + 1, ..., d_j$. So, we know that $a_{i(i>k_j),j}(A^{b_j^*}X^*) = 0$. This implies that $a_{i(i>k_j),j}(Y_j^*) = 0$. Thus, we have $Y_j^* \in S_{k_j}^j$. Similar analysis can be applied to the remaining loop conditions. Setting $b^* = max\{b_j^*\}_{j=1}^s$, we have $A^{b^*}X^* \in \bigcap_{j=1}^{s} S_{k_j}^j$. This completes the proof of the theorem. □

Remark 4. It is easy to see that any solution of $S_{k_1}^1 \bigcap S_{k_2}^2 \bigcap \cdots \bigcap S_{k_s}^s$ is a non-terminating point on which Program $\mathfrak{J}_1'$ does not terminate.

Now, let us consider the general program $\mathfrak{J}_1$ in which not all of the eigenvalues of A are equal.

For clarity, we restate the main idea due to Tiwari: Firstly, compute expression $x_i'(n)$ of every program variable x_i' after n iterations; Secondly, substitute $x_i'(n)$ for x_i' in loop conditions and obtain new conditions denoted by $Cond(n, X)$; Thirdly, guess a dominant term from every condition, and generate some constraints (i.e. semi-algebraic systems); Finally, get a witness to nontermination by solving these derived constraints if the solution set is not empty.

Next, we will present a recursive procedure to the termination of Program $\mathfrak{J}_1$. Different from Tiwari's method, in our recursive procedure, one does not need to guess a dominant term, since the new method can reduce the termination of Program $\mathfrak{J}_1$ to that of Program $\mathfrak{J}_0$ and $\mathfrak{J}_1'$, whose termination has been proven to be decidable by Corollary 2 and 3. This enables us to get at least one non-terminating point by solving semi-algebraic systems. To determine if Program $\mathfrak{J}_1$ is non-terminating is equivalent to determine if there exists a point $X^* \in \mathbb{R}^r$, such that for all $n \in \mathbb{Z}_{\geq 0}$, $Cond(n, X^*) > 0$. To do this, we first need to get the expression of $Cond(n, X)$.

After n iterations, the loop conditions of $\mathfrak{J}_1$ can be written as

$$Cond(n, X) = \begin{pmatrix} b_1^1J_1^n\mathbf{x}_1 + b_2^1J_2^n\mathbf{x}_2 + \cdots + b_m^1J_m^n\mathbf{x}_m \\ b_1^2J_1^n\mathbf{x}_1 + b_2^2J_2^n\mathbf{x}_2 + \cdots + b_m^2J_m^n\mathbf{x}_m \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_1^sJ_1^n\mathbf{x}_1 + b_2^sJ_2^n\mathbf{x}_2 + \cdots + b_m^sJ_m^n\mathbf{x}_m \end{pmatrix}$$

If $\lambda_i = \lambda_j$, then combine $b_i^vJ_i^n\mathbf{x}_i, b_j^vJ_j^n\mathbf{x}_j$ into single term, for all $v = 1, ..., s$. Collecting all "like" terms in $Cond(n, X)$, we can rewrite $Cond(n, X)$ as

$$Cond(n, X) = \begin{pmatrix} \lambda_1^n p_{11}(n, \mathbf{w}_1) + \lambda_2^n p_{12}(n, \mathbf{w}_2) + \cdots + \lambda_t^n p_{1t}(n, \mathbf{w}_t) \\ \lambda_1^n p_{21}(n, \mathbf{w}_1) + \lambda_2^n p_{22}(n, \mathbf{w}_2) + \cdots + \lambda_t^n p_{2t}(n, \mathbf{w}_t) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_1^n p_{s1}(n, \mathbf{w}_1) + \lambda_2^n p_{s2}(n, \mathbf{w}_2) + \cdots + \lambda_t^n p_{st}(n, \mathbf{w}_t) \end{pmatrix}$$

Where, $\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_t$ is a partition of the variable vector X, i.e. $X = \mathbf{w}_1 \cup \mathbf{w}_2 \cup ... \cup \mathbf{w}_t$, and $t \leq m$. Assume $\lambda_1 > \lambda_2 > \cdots > \lambda_t > 0$, and let $Jord_j(n, \mathbf{w}_j) = \lambda_j^n (p_{1j}(n, \mathbf{w}_j), p_{2j}(n, \mathbf{w}_j), \ldots, p_{sj}(n, \mathbf{w}_j))^T, j \in \overline{[1, t]}$. Then the condition $Cond(n, X)$ can be expressed equivalently as

$$Cond(n, X) \triangleq \\ Jord_1(n, \mathbf{w}_1) + Jord_2(n, \mathbf{w}_2) + \cdots + Jord_t(n, \mathbf{w}_t). \quad (4)$$

Denote by $J_{j_1,...,j_{s_j}}$, $j \in \overline{[1, t]}$, a diagonal matrix corresponding to $Jord_j$, in which the diagonal entries are s_j Jordan blocks having the same eigenvalue. If the geometric multiplicity of every eigenvalue of A is 1, then we have $J_{j_1,...,j_{s_j}} = J_j$. For example, if there exist only i and j such that $\lambda_i = \lambda_j$, then for all $v = 1, ..., s$, combine $b_i^v J_i^n \mathbf{x}_i, b_j^v J_j^n \mathbf{x}_j$ into single term, say $\bar{b}^v J^n_{w_1,...,w_{s_w}} \bar{x}$, where $\bar{b}^v = (b_i^v, b_j^v)$, $J_{w_1,...,w_{s_w}} = diag(J_i, J_j)$, and $\bar{x} = (\mathbf{x}_i^T, \mathbf{x}_j^T)^T$. Thus, in $Cond(n, X)$, $Jord_w = (\bar{b}^1 J^n_{w_1,...,w_{s_w}} \bar{x}, \bar{b}^2 J^n_{w_1,...,w_{s_w}} \bar{x}, ..., \bar{b}^s J^n_{w_1,...,w_{s_w}} \bar{x})^T$. It is easy to see that $Jord_w$ can be regarded as the expression after the n-th iteration of loop condition in the following loop:

$$while\ \ ((\bar{b}^1, \bar{b}^2, ..., \bar{b}^s)^T \bar{x} > 0)\ \ \{\bar{x} := J_{w_1,...,w_{s_w}} \bar{x}\}.$$

And the termination of this loop corresponding to $Jord_w$ can be determined by Corollary 2 and 3. Also, each component of $Jord_w$ can be regarded as the expression after the n-th iteration of loop condition in a certain loop. Since $Cond(n, X)$ (*or* $Jord_i$) is exactly corresponding to a certain loop, in the following, we say $Cond(n, X)$ (*or* $Jord_i$) is non-terminating, if there exists a point such that for all $n \in \mathbb{Z} \geq 0$, $Cond(n, X)$(*or* $Jord_i > 0$). Such a point is called an non-terminating point of $Cond(n, X)$ (*or*$Jord_i$). We also can say $Cond(n, X)$(*or* $Jord_i$) is non-terminating on such a point. Otherwise, if such a point does not exist, we say $Cond(n, X)$ (*or* $Jord_i$) is terminating for any input or terminating.

Definition 1. With the above notations. We say $Jord_1 \succ Jord_2 \succ \cdots \succ Jord_t$, if $\lambda_1 > \lambda_2 > \cdots > \lambda_t$.

According to Definition 1, $Jord_i$ grows faster than $Jord_j$ for $i < j$. Note that the termination of each $Jord_i$, for $i = 1, ..., t$, is decidable by Corollary 2 and 3, since it is exactly the expression after n iterations of loop condition of a certain program such as $\mathfrak{J}_0$ or $\mathfrak{J}'_1$. So, If there exists some i and $\mathbf{w}_i^*$, such that $Jord_i(n, \mathbf{w}_i^*) > 0$ holds for all $n \in \mathbb{Z}_{\geq 0}$, then there must exist one point $X^* = (0, ..., 0, \mathbf{w}_i^*, 0, ..., 0)^T$, such that $Cond(n, X^*) > 0$ holds for all $n \in \mathbb{Z}_{\geq 0}$, i.e., $Cond(n, X)$ is non-terminating on X^*. The below theorem suggests that under proper condition, one can determine the termination of $Cond$ by checking that of a certain $Jord_i$. Some notations are given first. For every p_{ij} in $Jord_j/\lambda_j^n$, denote by $\mathbf{coe}(p_{ij}) = \mathbf{coeffs}(p_{ij}, n)$ the coefficient set of polynomial p_{ij} w.r.t. n, and let $\mathbb{V}_R(\mathbf{coe}(p_{ij})) = \{\mathbf{w}_j \in R^{|\mathbf{w}_j|} :\ e(\mathbf{w}_j) = 0,$ *for all* $e \in \mathbf{coe}(p_{ij})\}$. Where $Jord_j/\lambda_j^n = (p_{1j}, ..., p_{sj})^T$. It is easy to see that the coefficient set $\mathbf{coe}(p_{ij})$ is a system of homogenous linear polynomials in $\mathbf{w}_j$. Denote $\mathbf{w}_{k+1..t} = (\mathbf{w}_{k+1}, ..., \mathbf{w}_t)$. Let

$$Cond_k(n, \mathbf{w}_{k+1..t}) \triangleq Jord_{k+1}(n, \mathbf{w}_{k+1}) + \cdots + Jord_t(n, \mathbf{w}_t),$$

and

$$Cond_k^{j_{1..v}}(n, \mathbf{w}_{k+1..t}) \triangleq Jord_{k+1}^{j_{1..v}}(n, \mathbf{w}_{k+1}) + \cdots + Jord_t^{j_{1..v}}(n, \mathbf{w}_t),$$

for $k = 0, ..., t-1$, where $j_{1..v} = \{j_1, ..., j_v\}$ be the set of row indices. Clearly, $Cond_k^{j_{1..v}}$ is composed of v components in $Cond_k$. If $j_{1..v} = \{1, 2, ..., s\}$, then $Cond_k^{j_{1..v}} = cond_k$. Let $Cond_0 = Cond$. In the following, for convenience, we say $Cond_k^{j_{1..v}}(n, \mathbf{w}_{k+1..t})$ is non-terminating, if there exists a point $\mathbf{w}^*_{k+1..t}$ such that for all $n \in \mathbb{Z}_{\geq 0}$, $Cond_k^{j_{1..v}}(n, \mathbf{w}^*_{k+1..t}) > 0$. $\mathbf{w}^*_{k+1..t}$ is called an non-terminating point of $Cond_k^{j_{1..v}}$. Also, we can say that $Cond_k^{j_{1..v}}$ is non-terminating on $\mathbf{w}^*_{k+1..t}$. Otherwise, if such a point does not exist, we say $Cond_k^{j_{1..v}}$ is terminating. Especially, for a given point $\mathbf{w}^*_{k+1..t}$, if there exists $n^0 \in \mathbb{Z}_{\geq 0}$, such that $Cond_k^{j_{1..v}}(n^0, \mathbf{w}^*_{k+1..t}) \leq 0$, we say $Cond_k^{j_{1..v}}(n, \mathbf{w}_{k+1..t})$ is terminating on $\mathbf{w}^*_{k+1..t}$.

The following results on the termination of $Cond_k^{j_{1..v}}$, for $k = 0, ..., t-2$, are established as follows.

Theorem 5. *With the above notion. If all of the element* $p_{i,k+1}$ *in* $Jord_{k+1}^{j_{1..v}}/\lambda_{k+1}^n$ *are not zero polynomials, and on each point such as*

$$\boldsymbol{w}_{k+1..t} = (\boldsymbol{w}^*_{k+1}, \boldsymbol{w}_{k+2}, \ldots, \boldsymbol{w}_t),$$

where $\boldsymbol{w}^*_{k+1} \in \bigcup_{i=j_1}^{j_v} \mathbb{V}_R(\mathbf{coe}(p_{i,k+1}))$, $Cond_k^{j_{1..v}}(n, \boldsymbol{w}_{\ k+1..t})$ *is terminating. Then there exists one point* $\boldsymbol{w}_{k+1..t}$, *such that for all* $n \in \mathbb{Z}_{\geq 0}$, $Cond_k^{j_{1..v}}(n, \boldsymbol{w}_{k+1..t}) > 0$ *if and only if there exists one point* $\boldsymbol{w}_{k+1}$ *such that* $Jord_{k+1}^{j_{1..v}}(n, \boldsymbol{w}_{k+1}) > 0$ *for all* $n \in \mathbb{Z}_{\geq 0}$.

Proof. The proof of sufficiency is easy. That is, if there exists a point $\mathbf{w}^0_{k+1}$ such that $Jord_{k+1}^{j_{1..v}}(n, \mathbf{w}^0_{k+1}) > 0$ for all $n \in \mathbb{Z}_{\geq 0}$, i.e., $Jord_{k+1}^{j_{1..v}}$ is non-terminating on $\mathbf{w}^0_{k+1}$, then there must exist one point $\mathbf{w}^0_{k+1,...,t} = (\mathbf{w}^0_{k+1}, 0, ..., 0)$ such that for all $n \in \mathbb{Z}_{\geq 0}$, $Cond_k^{j_{1..v}}(n, \mathbf{w}^0_{k+1..t}) > 0$, i.e., $Cond_k^{j_{1..v}}$ is non-terminating on $\mathbf{w}^0_{k+1..t}$. Next, we prove the necessity by contradiction. Assume that there does not exist a point $\mathbf{w}_{k+1} \in \mathbb{R}^{|\mathbf{w}_{k+1}|}$, such that $Jord_{k+1}^{j_{1..v}}(n, \mathbf{w}_{k+1}) > 0$ for all $n \in \mathbb{Z}_{\geq 0}$. First, we know that there exists one point, say $\mathbf{w}^0_{k+1..t} = (\mathbf{w}^0_{k+1}, \mathbf{w}^0_2, \ldots, \mathbf{w}^0_t)^T$, such that

$$Cond_k^{j_{1..v}}(n, \mathbf{w}^0_{k+1..t}) > 0,$$

for all $n \in \mathbb{Z}_{\geq 0}$. And all points on which $Cond_k^{j_{1..v}}(n, \mathbf{w}_{k+1..t}) > 0$ for all $n \in \mathbb{Z}_{\geq 0}$, can be divided into two types:

1) $\mathbf{w}_{k+1} \in \bigcup_{i=j_1}^{j_v} \mathbb{V}_R(\mathbf{coe}(p_{i,k+1}))$.

2) $\mathbf{w}_{k+1} \notin \bigcup_{i=j_1}^{j_v} \mathbb{V}_R(\mathbf{coe}(p_{i,k+1}))$.

By the condition of the theorem, we know $\mathbf{w}^0_{k+1..t}$ belongs only to the second type, i.e. for all $i \in \overline{[1, s]}$, $p_{i,k+1}(\mathbf{w}^0_{k+1}) \neq 0$. Since $Cond_k^{j_{1..v}}(n, \mathbf{w}^0_{k+1..t}) > 0$ for all $n \in \mathbb{Z}_{\geq 0}$, after K iterations, say $K = max\{K_i\}_{i=1}^s$, the sign of $Cond_k^{j_{1..v}}$ is determined only by the sign of $Jord_{k+1}^{j_{1..v}}$. Where K_i denotes some integer such that the sign of the i-th expression in $Cond_k^{j_{1..v}}$ is determined only by the sign of $p_{i,k+1}$ in $Jord_{k+1}^{j_{1..v}}$ when $n > K_i$. Now, let

$$\mathbf{w}'_{k+1..t} = (diag(J_{k+1_1,...,k+1_{s_{k+1}}}, ..., J_{t_1,...,t_{s_t}}))^K \cdot \mathbf{w}^0_{k+1..t} \\ = (\mathbf{w}'_{k+1}, ..., \mathbf{w}'_t).$$

It follows that $Jord_{k+1}^{j_{1..v}}\ (n, \mathbf{w}'_{k+1}) > 0$ holds for all $n \in \mathbb{Z}_{\geq 0}$. This contradicts the assumption. Hence, this contradiction completes the proof. □

When $Jord_{k+1}^{j_{1..v}}(n, \mathbf{w}_{k+1}) \equiv 0$, we have $Cond_k^{j_{1..v}}(n, \mathbf{w}_{\ k+1..t}) = Cond_{k+1}^{j_{1..v}}(n, \mathbf{w}_{\ k+2..t})$. Then, checking the termination of $Cond_k^{j_{1..v}}(n, \mathbf{w}_{\ k+1..t})$ is equivalent to checking that of $Cond_{k+1}^{j_{1..v}}$. But, if some $b_u^v s$ in the condition matrix B are zero, then some of the components of $Jord_{k+1}^{J_{1..v}}(n,\ \mathbf{w}_{k+1})$ must be zero. Especially, some of the components of $Jord_{k+1}^{J_{1..v}}(n, \mathbf{w}_{k+1}^*)$ are zero in the case when $Cond_k^{J_{1..v}}(n,\ \mathbf{w}_{k+1..t})$ is evaluated at such points $\mathbf{w}_{k+1}^* \in \mathbb{V}_R(\mathbf{coe}(p_{i,k+1}))$. Note that because $\mathbf{coe}(p_{i,k+1})$ is a homogeneous linear system, it either has positive-dimensional solution (i.e. some of variables, say $x_{i_1}, ..., x_{i_l}$, can always be expressed as linear combinations of the other variables, i.e., $x_{i_j} = \sum_{x_\alpha \in \mathbf{w}_{k+1}/\{x_{i_1},...,x_{i_l}\}} c_\alpha x_\alpha$, for $j = 1, ..., l$. In this case, $\mathbf{w}_{k+1}^*$ denotes the one obtained by substituting $x_{i_j} = \sum_{x_\alpha \in \mathbf{w}_{k+1}/\{x_{i_1},...,x_{i_l}\}} c_\alpha x_\alpha$ into $\mathbf{w}_{k+1}$. or has the solution $\mathbf{w}_{k+1}^* = 0$. Unless otherwise specified, in the following, $Cond_k^{j_{1..v}}$, $Jord_{k+1}^{j_{1..v}}$ and $p_{i,k+1}$ denote the expressions before and after $Cond_k^{j_{1..v}}(n, \mathbf{w}_{k+1..t})$, $Jord_{k+1}^{j_{1..v}}(n, \mathbf{w}_{k+1})$ and $p_{i,k+1}(n, \mathbf{w}_{k+1})$ are evaluated at $\mathbf{w}_{k+1}$, respectively. In our recursive procedure, we usually need to determine the termination of

$$Cond_k^{j_{1..v}}(n, \mathbf{w}_{k+1..t})|_{\mathbf{w}_{k+1}=\mathbf{w}_{k+1}^*}.$$

If $\mathbf{w}_{k+1}^* = 0$, then $Cond_k^{j_{1..v}}(n, \mathbf{w}\ _{k+1..t})|_{\mathbf{w}_{k+1}=0} = Cond_{k+1}^{j_{1..v}}(n, \mathbf{w}\ _{k+2..t})$. Otherwise, the remaining variables

$$x_{i'_j} \in D_{var} = \mathbf{w}_{k+1}/\{x_{i_1}, ..., x_{i_l}\}$$

will remain in the nonzero components of $Jord_{k+1}^{j_{1..v}}(n, \mathbf{w}_{k+1}^*)$. Since

$$Jor_{k+1}^{J_{1..v}}(n, \mathbf{w}_{k+1})|_{\mathbf{w}_{k+1}=\mathbf{w}_{k+1}^*},$$

and $Jord_{k+1}^{J_{1..v}}(n,\ \mathbf{w}_{k+1})$ have at least one component equal to zero respectively, we can use the following corollary to check the termination of the corresponding $Cond_k^{j_{1..v}}$. Some notations are given first.

Let $z_{1..u} = \{z_1, ..., z_u\} = \{i \in J_{1..v} :\ p_{i,k+1} \equiv 0, p_{i,k+1} \in Jord_{k+1}^{j_{1..v}}\}$. Then, for any $i \in j_{1..v}/z_{1..u}$, $p_{i,k+1}$ is a nonzero polynomial. Let $Jord_{k+1}^{j_{1..v}/z_{1..u}}$ be a vector composed of all the nonzero components in $Jord_{k+1}^{j_{1..v}}$. Where, we denote by $Jord_{k+1}^{j_{1..v}/z_{1..u}}$ $Jord_{k+1}^{j_{1..v}/z_{1..u}}(n, \mathbf{w}_{k+1})$ and $Jord_{k+1}^{j_{1..v}/z_{1..u}}(n, \mathbf{w}_{k+1}^*)$. For convenience, we say $Jord_{k+1}^{j_{1..v}/z_{1..u}}$ is non terminating, if there exists a point on which $Jord_{k+1}^{j_{1..v}/z_{1..u}} > 0$ for all $n \in \mathbb{Z}$.

Corollary 4. *With the above notion. If on each point such as*

$$\boldsymbol{w}_{k+1,...,t} = (\boldsymbol{w}_{k+1}^*, \boldsymbol{w}_{k+2}, \ldots, \boldsymbol{w}_t)$$

where $\boldsymbol{w}_{k+1}^* \in \bigcup_{i \in j_{1..v}/z_{1..u}} \mathbb{V}_R(\mathbf{coe}(p_{i,k+1}))$, $Cond_k^{j_{1..v}}$ *is terminating. Then,* $Jord_{k+1}^{j_{1..v}/z_{1..u}}$ *is terminating implies that* $Cond_k^{j_{1..v}}$ *is terminating.*

Proof. The proof is very simple and very similar to that of Theorem 5. We omit it here. ☐

Corollary 4 gives a sufficient but not necessary condition on which $Cond_k^{j_{1..v}}$ is terminating. However, if the condition of Corollary 4 is satisfied and $Jord_{k+1}^{j_{1..v}/z_{1..u}}$ is non-terminating, then we have to check the termination of $Cond_{k+1}^{z_{1..u}}$,

$$Cond_{k+1}^{z_{1..u}} = Jord_{k+2}^{z_{1..u}} + Jord_{k+3}^{z_{1..u}} + \cdots + Jord_t^{z_{1..u}}.$$

The reason is that,$Jord_{k+1}^{j_{1..v}/z_{1..u}}$ is non-terminating only implies that $Cond_k^{j_{1..v}/z_{1..u}}$ is non-terminating and can not imply $Cond_k^{j_{1..v}}$ is non-terminating, since $Cond_k^{j_{1..v}}$ is non-terminating if and only of all of the components of $Cond_k^{j_{1..v}}$ are non-terminating. Therefore, when it happens, we need to check the termination of $Cond_{k+1}^{z_{1..u}}$. This can be done by Theorem 5, if there are not zero polynomials in $Jord_{k+2}^{z_{1..u}}$. Otherwise, Corollary 4 will be used again.

Let us consider $Cond_k^{j_{1..v}}$. If all of the components of $Cond_k^{j_{1..v}}$ are not zero polynomials, then Theorem 5 can be used to decide the termination of $Cond_k^{j_{1..v}}$. According to Theorem 5, we need to first determine if $Cond_k^{j_{1..v}}$ terminates on those points $\mathbf{w}_{k+1,...,t} = (\mathbf{w}_{k+1}^*, \mathbf{w}_{k+2}, \ldots, \mathbf{w}_t)$, where $\mathbf{w}_{k+1}^* \in \bigcup_{i=j_1}^{j_v} \mathbb{V}_R(\mathbf{coe}(p_{i,k+1}))$. If $Cond_k^{j_{1..v}}$ terminates on all such points, checking if $Cond_k^{j_{1..v}}$ is nonterminating is equivalent to checking if $Jord_{k+1}^{j_{1..v}}$ is nonterminating by Theorem 5. And the latter is decidable by Corollary 2 or 3, since $Jord_{k+1}^{j_{1..v}}$ is the expression after the n-th iteration of loop condition in a certain loop such as Program $\mathfrak{J}_0$ or $\mathfrak{J}_1'$. Therefore, the key step is to decide whether the condition of Theorem 5 is satisfied or not. To do this, for each $p_{i,k+1}$, we need to compute $\mathbb{V}_R(\mathbf{coe}(p_{i,k+1}))$. Usually, the linear system $\mathbf{coe}(p_{i,k+1})$ has infinitely many solutions. Thus, some of variables, say $x_{i_1}, ..., x_{i_l}$, can always be expressed as linear combinations of the other variables, i.e., $x_{i_j} = \sum_{x_\alpha \in \mathbf{w}_{k+1}/\{x_{i_1},...,x_{i_l}\}} c_\alpha x_\alpha$, for $j = 1, ..., l$. Denote by $\mathbf{w}_{k+1}^*$ the expression of $\mathbf{w}_{k+1}$ after substitute each $x_{i_j} = \sum_{x_\alpha \in \mathbf{w}_{k+1}/\{x_{i_1},...,x_{i_l}\}} c_\alpha x_\alpha$ into $\mathbf{w}_{k+1}$. Clearly, $\mathbf{w}_{k+1}^* \in \mathbb{V}_R(\mathbf{coe}(p_{i,k+1}))$. Then, substituting $\mathbf{w}_{k+1}^*$ into $Cond_k^{j_{1..v}}$, we can eliminate the variables x_{i_j}, $i_j \in \{i_1, ..., i_l\}$, in $p_{j \neq i,k+1}$, and we have $p_{i,k+1} \equiv 0$. In general, if $\mathbf{w}_{k+1}^* \in \bigcap_{i=j_1}^{j_v} \mathbb{V}_R(\mathbf{coe}\ (p_{i,k+1}))$, then after substitution, $Jord_{k+1}^{j_{1..v}}(n, \mathbf{w}_{k+1}^*) \equiv 0$, i.e., $Jord_{k+1}^{j_{1..v}}(n, \mathbf{w}_{k+1}^*)$ does not contain any variable. If not, then the variables $x_{i'_j}$, $x_{i'_j} \in D_{var}$, will remain in the nonzero components in $Jord_{k+1}^{j_{1..v}}(n, \mathbf{w}_{k+1}^*)$. Let

$$\begin{aligned} Cond_k^{j_{1..v}}|_{\mathbf{w}_{k+1}=\mathbf{w}_{k+1}^*} =& Jord_{k+1}^{j_{1..v}}(n, \mathbf{w}_{k+1}^*) + \\ &\cdots + Jord_t^{j_{1..v}}(n, \mathbf{w}_t) \end{aligned}$$

and $\mathbf{w}_{k+1..t}^{p_{i,k+1}} = (\mathbf{w}_{k+1}^*, \mathbf{w}_{k+2}, ..., \mathbf{w}_t)^T$. Next, we need to check if $Cond_k^{j_{1..v}}(n, \mathbf{w}_{k+1})$ terminates on $\mathbf{w}_{k+1..t}^{p_{i,k+1}}$. This is equivalent to check if $Cond_k^{j_{1..v}}|_{\mathbf{w}_{k+1}=\mathbf{w}_{k+1}^*}$ is terminating. If $Jord_{k+1}^{j_{1..v}}(n, \mathbf{w}_{k+1}^*) \equiv 0$ (this case can happen when $\mathbf{w}_{k+1}^* \in \bigcap_{i=j_1}^{j_v} \mathbb{V}_R(\mathbf{coe}(p_{i,k+1})))$, then we have

$$Cond_k^{j_{1..v}}|_{\mathbf{w}_{k+1}=\mathbf{w}_{k+1}^*} = Cond_{k+1}^{j_{1..v}}(n, \mathbf{w}_{k+2..t}).$$

Thus, we can proceed to apply Theorem 5 or its corollay to check the termination of $Cond_k^{j_{1..v}}(n, \mathbf{w}_{k+2..t})$. Otherwise, we can apply Corollary 4 to check the termination of $Cond_k^{j_{1..v}}|_{\mathbf{w}_{k+1}}$, since at least the i-th component of $Jord_{k+1}^{j_{1..v}}(n, \mathbf{w}_{k+1}^*)$ is a zero polynomial and the variables x_{i_j}, $x_{i_j} \in \{x_{i_1}, ..., x_{i_l}\}$, remain in the nonzero components of $Jord_{k+1}^{j_{1..v}}(n, \mathbf{w}_{k+1}^*)$. The whole process is recursive. That is, when applying Theorem 5 or its corollary to determine the termination of $Cond_k^{j_{1..v}}$ and $Jord_{k+1}^{j_{1..v}}$, we always need to check whether the conditions of Theorem 5 or its corollary is satisfied first. The decision process described as above is also terminating since $t < \infty$.

THEOREM 6. *The termination of Program $\mathfrak{J}_1$ is decidable over $\mathbb{R}^r$.*

PROOF. Theorem 6 follows immediately form Theorems 10, Corollary 4 and the above arguments. $\square$

4. EXAMPLES

In the section, we take an example to illustrate our methods established as above.

Example 3. Consider the below linear program

$$Q_6: \quad while \;\; (BX>0)\;\; \{X:=AX\}.$$

Where

$$B=\begin{pmatrix} -1 & 0 & 6 & -3 & 0 & 7 & -2 \\ 2 & 4 & -3 & 2 & 1 & -11 & 0 \end{pmatrix}$$

$$A=\begin{pmatrix} J_1(\lambda_1) & & \\ & J_2(\lambda_2) & \\ & & J_3(\lambda_3) \end{pmatrix}$$

J_1, J_2 and J_3 are $3\times 3, 2\times 2$ and 2×2 Jordan blocks, respectively. Let $\lambda_1=\lambda_2=3, \lambda_3=2$.

The loop condition after n iterations can be written as:

$$Cond_0=\begin{pmatrix} b_1^1 J_1^n \mathbf{x}_1 + b_2^1 J_2^n \mathbf{x}_2 + b_3^1 J_3^n \mathbf{x}_3 \\ b_1^2 J_1^n \mathbf{x}_1 + b_2^2 J_2^n \mathbf{x}_2 + b_3^2 J_3^n \mathbf{x}_3 \end{pmatrix}. \quad (5)$$

Combining like terms, we rewrite $Cond_0$ as:

$$Cond_0=\begin{pmatrix} 3^n p_{11}(n,\mathbf{w}_1)+2^n p_{12}(n,\mathbf{w}_2) \\ 3^n p_{21}(n,\mathbf{w}_1)+2^n p_{22}(n,\mathbf{w}_2) \end{pmatrix}. \quad (6)$$

Where $\mathbf{w}_1=(x_1,x_2,x_3,x_4,x_5)^T, \mathbf{w}_2=(x_6,x_7)^T$, $p_{11}=(-\frac{x_3 n^2}{18}+(\frac{x_3}{18}-\frac{x_2}{3}-x_5)n-x_1-3x_4+6x_3)$, $p_{12}=(\frac{7nx_7}{2}+7x_6-2x_7)$, $p_{21}=\frac{x_3 n^2}{9}+(\frac{11x_3}{9}+\frac{2x_2}{3}+\frac{2x_5}{3})n+2x_1+4x_2+2x_4-3x_3+x_5$, and $p_{22}=-11x_6-\frac{11nx_7}{2}$. Let $Jord_1=3^n(p_{11},p_{21})^T$, $Jord_2=2^n(p_{12},p_{22})^T$. Define

$$Cond_0(n,\mathbf{w}_{1..2})=Cond_0^{\{1,2\}}(n,\mathbf{w}_{1..2}) \triangleq Jord_1^{\{1,2\}}(n,\mathbf{w}_1)+Jord_2^{\{1,2\}}(n,\mathbf{w}_2).$$

Since there is no zero polynomial in $Jord_1$, we can determine if there exists a point such that for all $n\in\mathbb{Z}_{\geq 0}$, $Cond_0(n,X)>0$ by Theorem 5. According to the precondition of Theorem 5, if $Cond_0(n,X)$ terminates on each point in set $\bigcup_{i=1}^2 \mathbb{V}_R(\mathbf{coe}(p_{i,1}))$, then checking if $Cond_0(n, \mathbf{w}_{1..t})$ is non-terminating is equivalent to checking if $Jord_1(n,\mathbf{w}_1)$ is non-terminating. Clearly, $Jord_1$ is exactly the expression after n iterations of the loop condition of the below program,

$$Q_7: \;\; while \;\; (B^{(1)}\mathbf{w}_1>0)\;\; \{\mathbf{w}_1:=A^{(1)}\mathbf{w}_1\}.$$

Where

$$B^{(1)}=\begin{pmatrix} -1 & 0 & 6 & -3 & 0 \\ 2 & 4 & -3 & 2 & 1 \end{pmatrix}, A^{(1)}=\begin{pmatrix} J_1 & \\ & J_2 \end{pmatrix}.$$

Therefore, Corollary 3 can be used to determine the termination of $Jord_1(n,\mathbf{w}_1)$, since Program Q_7 is a program such as $\mathfrak{J}_1'$. Now, the key step is to check if the precondition of Theorem 5 is satisfied. Since $\mathbf{coe}(p_{11})=\{-\frac{x_3}{18},\frac{x_3}{18}-\frac{x_2}{3}-x_5,-x_1-3x_4+6x_3\}$ and $\mathbf{coe}(p_{21})=\{\frac{x_3}{9},\frac{11x_3}{9}+\frac{2x_2}{3}+\frac{2x_5}{3},2x_1+4x_2+2x_4-3x_3+x_5\}$, by solving the above systems of linear equations, we get

$$\mathbb{V}_R(\mathbf{coe}(p_{11}))=\{\mathbf{w}_1\in\mathbb{R}^5: x_1=-3x_4,x_2=-3x_5,x_3=0,x_4=x_4,x_5=x_5\}$$

and

$$\mathbb{V}_R(\mathbf{coe}(p_{21}))=\{\mathbf{w}_1\in\mathbb{R}^5: x_1=\frac{3}{2}x_5-x_4,x_2=-x_5,x_3=0,x_4=x_4,x_5=x_5\}.$$

Let $\mathbf{w}_1^{(*1)}=(-3x_4,-3x_5,0,x_4,x_5)^T$ and let $\mathbf{w}_1^{(*2)}=(\frac{3}{2}x_5-x_4,-x_5,0,x_4,x_5)^T$. And $D_{var}=(x_4,x_5)$. Substituting $\mathbf{w}_1=\mathbf{w}_1^{(*1)}$ and $\mathbf{w}_1=\mathbf{w}_1^{(*2)}$ into $Cond_0$ respectively, we get

$$Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*1)}}=\begin{pmatrix} 0 + 2^n p_{12}>0 \\ 3^n p_{21}(n,\mathbf{w}_1^{(*1)})+2^n p_{22}>0 \end{pmatrix} \quad (7)$$

and

$$Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*2)}}=\begin{pmatrix} 3^n p_{11}(n,\mathbf{w}_1^{(*2)})+2^n p_{12}>0 \\ 0 + 2^n p_{22}>0 \end{pmatrix} \quad (8)$$

To check if $Cond_0^{\{1,2\}}$ satisfies the condition of Theorem 5 is equivalent to check if both of $Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*1)}}$ and $Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*2)}}$ are terminating. Next, we just consider $Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*1)}}$, and similar analysis can be applied to the termination of $Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*2)}}$. Define

$$Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*1)}}=Jord_1^{\{1,2\}}(n,\mathbf{w}_1^{(*1)})+Jord_2^{\{1,2\}}(n,\mathbf{w}_2)$$

Where

$$Jord_1^{\{1,2\}}(n,\mathbf{w}_1^{(*1)})=(0,3^n p_{21}(n,\mathbf{w}_1^{(*1)}))^T = (0,3^n p_{21}^{(11)}(n,\mathbf{w}_1))^T$$

and $p_{21}^{(11)}(n,\mathbf{w}_1)=-\frac{4}{3}nx_5-4x_4-11x_5$. Since the first component of $Jord_1^{\{1,2\}}(n,\mathbf{w}_1^{(*1)})$ is a zero polynomial, we can only use Corollary 4 to determine the termination of $Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*1)}}$. Let $z_{1..1}=\{1\}$. Delete the component equal to zero, and get $Jord_1^{\{1,2\}/z_{1..1}}=3^n p_{21}^{(11)}(n,\mathbf{w}_1)$. Corollary 4 tells us that if $Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*1)}}$ terminates on all points in set $\mathbb{V}_R(\mathbf{coe}(p_{21}(n,\mathbf{w}_1^{(*1)})))(=\mathbb{V}_R(\mathbf{coe}(p_{21}^{(11)})))$, then $Jord_1^{\{1,2\}/z_{1..1}}$ terminates implies that $Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*1)}}$ terminates. By computing, we have

$$\mathbb{V}_R(\mathbf{coe}(p_{21}^{(11)}))=\{\mathbf{w}_1\in\mathbb{R}^5, x_1=x_1,x_2=x_2,x_3=x_3, x_4=x_5=0\}.$$

Let $\mathbf{w}_1^{(11)}=(x_1,x_2,x_3,0,0)^T$. Substituting $\mathbf{w}_1^{(11)}$ into $Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*1)}}$, we get

$$Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*11)}}=Cond_1^{\{1,2\}}(n,\mathbf{w}_2)\triangleq Jord_2^{\{1,2\}}(n,\mathbf{w}_2)$$

where $\mathbf{w}_1^{(*11)}=\mathbf{w}_1^{(*1)}\cap\mathbf{w}_1^{(11)}=(0,0,0,0,0)^T$. Thus, if $Cond_1^{\{1,2\}}(n,\mathbf{w}_2)$ terminates, then it follows that $Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*1)}}$ terminates on all points in set $\mathbb{V}_R(\mathbf{coe}(p_{21}^{(11)}))$. Otherwise, if $Cond_1^{\{1,2\}}(n,\mathbf{w}_2)$ does not terminate on a point $\mathbf{w}_2=\mathbf{w}_2^*$, then $Cond_0^{\{1,2\}}(n,\mathbf{w}_{1..2})$ must not determinate on the point $(0,0,0,0,0,\mathbf{w}_2^*)^T$. Once this happens, the recursive decision process terminates. Next, we will consider the termination of $Cond_1^{\{1,2\}}(n,\mathbf{w}_2)$. Since $Jord_2^{\{1,2\}}(n,\mathbf{w}_2)$ is the expression after n iterations of the loop condition of the program:

$$Q_8: \quad while(B^{(2)}\mathbf{w}_2>0)\;\; \{\mathbf{w}_2:=A^{(2)}\mathbf{w}_2\}.$$

Where

$$B^{(2)} = \begin{pmatrix} 7 & -2 \\ -11 & 0 \end{pmatrix}, \quad A^{(2)} = J_3.$$

By Theorem 3, we see that Q_8 is terminating, since $7 \times (-11) < 0$. Certainly, we can also use Corollary 2 to get the same result. By Corollary 2, construct the semi-algebraic sets:

$$s_1^1 = \{(x_6, x_7) : \frac{7}{2}x_7 > 0, 7x_6 - 2x_7 > 0\} \quad (9)$$

$$s_0^1 = \{(x_6, x_7) : \frac{7}{2}x_7 = 0, 7x_6 - 2x_7 > 0\} \quad (10)$$

$$s_1^2 = \{(x_6, x_7) : -\frac{11}{2}x_7 > 0, -11x_6 > 0\} \quad (11)$$

$$s_0^2 = \{(x_6, x_7) : -\frac{11}{2}x_7 = 0, -11x_6 > 0\} \quad (12)$$

It is easy to see that the semi-algebraic sets $s_1^1 \bigcap s_1^2, s_1^1 \bigcap s_0^2$, $s_0^1 \bigcap s_1^2, s_0^1 \bigcap s_0^2$ are all empty. This implies Q_8 is indeed terminating. This implies that $Cond_1^{\{1,2\}}(n, \mathbf{w}_2)$ is terminating, too. Therefore, $Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*1)}}$ terminates on all points in set $\mathbb{V}_R(\mathbf{coe}(p_{21}^{(11)}))$. So, $Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*1)}}$ satisfies the condition of Corollary 4. By Corollary 4, if $Jord_1^{\{1,2\}/z_{1..1}}$ terminates, then $Cond_0^{\{1,2\}}|_{\mathbf{w}_1=\mathbf{w}_1^{(*1)}}$ terminates. Next, we need to decide if $Jord_1^{\{1,2\}/z_{1..1}}$ is terminating. Since $Jord_1^{\{1,2\}/z_{1..1}} = 3^n p_{21}(n, \mathbf{w}_1^{(*1)}) = 3^n p_{21}^{(11)}(n, \mathbf{w}_1)$ $= 3^n(-\frac{4}{3}nx_5 - 4x_4 - 11x_5)$, it is easy to see that when $x_4 = x_5 = -1$, the inequality $3^n(-\frac{4}{3}nx_5 - 4x_4 - 11x_5) > 0$ holds for all $n \in \mathbb{Z}_{\geq 0}$. So, $Jord_1^{\{1,2\}/z_{1..1}}$ does not terminate on a point $(-1, -1)^T$. We also present a more general method to determine the termination of $Jord_1^{\{1,2\}/z_{1..1}}$ by checking if a semi-algebraic set is empty. Our main idea is: First, since $3^n p_{21}(n, \mathbf{w}_1)$ can be regarded as the expression of loop condition after n iterations of Program Q_9, we construct a subset S_{Q_9} of the set of nonterminating points of Q_9 by Theorem 4,

$$Q_9: \quad while\ (B^{(3)}\mathbf{w}_1 > 0)\ \{\mathbf{w}_1 := A^{(3)}\mathbf{w}_1\}.$$

Where

$$B^{(3)} = (2, 4, -3, 2, 1), \quad A^{(3)} = \begin{pmatrix} J_1 & \\ & J_2 \end{pmatrix}$$

Then, substitute $\mathbf{w}_1 = \mathbf{w}_1^{(*1)}$ into S_{Q_9}, and get a semi-algebraic set S'_{Q_9}. If S'_{Q_9} is empty, then $Jord_1^{\{1,2\}/z_{1..1}}$ terminates; Otherwise, $Jord_1^{\{1,2\}/z_{1..1}}$ does not terminate on any point in S'_{Q_9}. In terms of Theorem 4, we get

$$\begin{aligned} S_{Q_9} = \{(x_1, ..., x_5) : &\frac{1}{9}x_3 > 0, \frac{11}{9}x_3 + \frac{2}{3}x_2 + \frac{2}{3}x_5 > 0, \\ &2x_1 + 4x_2 + 2x_4 - 3x_3 + x_5 > 0\} \\ \bigcup\{(x_1, ..., x_5) : &\frac{1}{9}x_3 = 0, \frac{11}{9}x_3 + \frac{2}{3}x_2 + \frac{2}{3}x_5 > 0, \\ &2x_1 + 4x_2 + 2x_4 - 3x_3 + x_5 > 0\} \\ \bigcup\{(x_1, ..., x_5) : &\frac{1}{9}x_3 = 0, \frac{11}{9}x_3 + \frac{2}{3}x_2 + \frac{2}{3}x_5 = 0, \\ &2x_1 + 4x_2 + 2x_4 - 3x_3 + x_5\} > 0. \end{aligned}$$

After substituting $\mathbf{w}_1 = \mathbf{w}_1^{(*1)}$ into S_{Q_9}, we have

$$\begin{aligned} S'_{Q_9} &= \{(x_1, ..., x_5) : 0 < 0, -\frac{4}{3}x_5 > 0, -4x_4 - 11x_5 > 0\} \\ &\bigcup \{(x_1, ..., x_5) : 0 = 0, -\frac{4}{3}x_5 > 0, -4x_4 - 11x_5 > 0\} \\ &\bigcup \{(x_1, ..., x_5) : 0 = 0, -\frac{4}{3}x_5 = 0, -4x_4 - 11x_5 > 0\}. \end{aligned}$$

Clearly, S'_{Q_9} is nonempty since $(x_4, x_5)^T = (-1, -1)^T \in S'_{Q_9}$. This implies that $Jord_1^{\{1,2\}/z_{1..1}}$ is indeed nonterminating. But, we can not draw a conclusion that $Cond_0^{\{1,2\}}$ is also non-terminating since $Cond_0^{\{1,2\}}$ is non-terminating if and only if all of the components of $Cond_0^{\{1,2\}}$ are non-terminating. Next, to decide the termination of $Cond_0^{\{1,2\}}$, we need to consider the termination of $Cond_1^{z_{1..1}} = Cond_1^{\{1\}}$ obtained by deleting the second component from $Cond_0^{\{1,2\}}$.

$$Cond_1^{\{1\}} = 2^n p_{12}(n, \mathbf{w}_2).$$

$Cond_1^{\{1\}}$ is the expression after n iterations of loop condition of the program:

$$Q_{10}: \quad while\ (B^{(4)}\mathbf{w}_2 > 0)\ \{\mathbf{w}_2 := A^{(4)}\mathbf{w}_2\}.$$

Where

$$B^{(4)} = \begin{pmatrix} 7 & -2 \end{pmatrix}, \quad A^{(4)} = J_3.$$

By Theorem 4, we can get a witness to non-termination of Q_{10}, say $\mathbf{w}_2^* = (1, 1)^T$. Substitute the values of x_4 and x_5 into $\mathbf{w}_1^{(*1)}$, and get $\mathbf{w}_1^{(01*)} = (3, 3, 0, -1, -1)^T$. Let $X^* = (\mathbf{w}_1^{(01*)}, \mathbf{w}_2^*)^T = (3, 3, 0, -1, -1, 1, 1)^T$. It immediately follows that X^* is a witness to non-termination of Program Q_6.

5. CONCLUSION

In this paper, we construct the subsets of sets of nonterminating points of two special classes of linear programs. By checking if the subsets are empty, we can determine the termination of them. Furthermore, a recursive decision method is presented to reduce the termination of a more general class of linear programs to the termination of the above two special class of linear programs. Different from the methods of Tiwari and Braverman, we do not need to guess a dominant term from every loop condition in our recursive algorithm.

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