Witness to Non-termination of Linear Programs

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Abstract

In his CAV 2004 paper, Tiwari has proved that, for a class of linear programs over the reals, termination is decidable. And Tiwari has shown that the termination of a linear program $\mathbf{P}_1$ whose assignment matrix $\widetilde{A}$ is not in the Jordan canonical form is equivalent to that of a linear program $\mathfrak{J}_1$, whose assignment matrix A is in the Jordan Canonical Form. In most cases, the method of Tiwari provides only a so-called N-nonterminating point. In this paper, we propose two new methods to decide whether Program $\mathbf{P}_1$ terminates or not over the reals. Our methods are based on the construction of a subset of the set NT of non-terminating points of Program $\mathfrak{J}_1$. Any point in such a subset is a witness to non-termination of Program $\mathfrak{J}_1$. Furthermore, it is shown that Program $\mathfrak{J}_1$ is non-terminating if and only if such a subset is nonempty. In terms of the property, the first method is given to check whether Program $\mathfrak{J}_1$ terminates or not. Different from the existing methods, the point obtained by our first method is a non-terminating point, rather than a N-nonterminating point. More importantly, such a subset is also proven to be $A^{\widehat{B}}$-invariant for some positive integer $\widehat{B}$. This enables us to check directly the termination of Program $\mathfrak{J}_1$ by verifying the satisfiability of finitely many quantified formulas over the reals. This suggests our second method for checking the termination of Program $\mathfrak{J}_1$.

Keywords: Linear Loops, Program Termination, Semi-algebraic Sets, Witness to Non-termination

1. Introduction

It is well known that guaranteeing software systems trustworthy is a grand challenge in theoretical computer science [1, 2, 3]. As one of the building blocks of automated program verification, termination analysis has attracted increasing interest in the recent years. However, the termination problem is undecidable in most cases. Therefore, most well-established work concentrates on the construction of well-founded ranking functions [4, 5, 6, 7, 8, 9]. Especially, Poldelski and Rybalchenko [9] first presented a complete method for the synthesis of linear ranking functions in 2004. But, it has been shown that the existence of ranking functions is just a sufficient (but not necessary) condition for guaranteeing the termination of loops. That is to say, one can construct an example of a loop that terminates but has no ranking function. Because of the reasons mentioned above, people pay attention to explore a decidable class of loops. For example, Tiwari [10] in 2004 showed that the termination of a linear loop program of the following shape is decidable over the reals, as follows:

$$\mathbf{P_1} \quad \textit{while} \ \ (\widetilde{B}X > 0) \ \ \{X := \widetilde{A}X\}$$

where $\widetilde{B} \in \mathbb{R}^{s\times L}$ is called the condition matrix, and $\widetilde{A} \in \mathbb{R}^{L\times L}$ is called the assignment matrix. This classic work shows us new insight on termination problem for programs. It has been pointed out that although a linear program may not be presented in this form, termination problem can always be reduced to this form by [10]. In Tiwari's method, the termination of $\mathbf{P}_1$ is reduced equivalently to that of $\mathfrak{J}_1$ whose assignment matrix is in the Jordan canonical form. In 2006, Braverman [11] generalized the work of Tiwari and proved that the loops of the above class is also decidable over the integers. But, the methods of Tiwari and Braverman do not consider how to get a witness to nontermination. Following their work, Xia et al. [12] considered the termination of a more general class of loops with nonlinear constraints and linear updates. They proved that under proper conditions, such loops were decidable over the reals. Since the decision procedure given by Tiwari depends on the computation of Jordan canonical forms, Yang et al. [13] p-resented a purely symbolic method to compute Jordan canonical forms. In addition, under the assumption that $\widetilde{A}$ is diagonalizable matrix and its all eigenvalues are real, Rebiha et al. studied the termination of $\mathbf{P}_1$ and presented a method of generating the set of N-nonterminating points of $\mathbf{P}_1$ in [14, 15]. Recently, Ouaknine et al. [16] show decidability of termination of simple linear loops over the integers under the assumption that the assignment matrix is diagonalizable. And the work of Ouaknine et al. is the first substantial advance on an open problem of Braverman [11].

In [17], we reconsider the same termination problem proposed and analyzed by Tiwari in 2004, and present a recursive algorithm for the termination of program $\mathbf{P}_1$. For clarity, we describe below the main ideas presented in [17] briefly. First, we reduce $\mathbf{P}_1$ to $\mathfrak{J}_1$ by the computation of the Jordan canonical form of the assignment matrix $\widetilde{A}$ of $\mathbf{P}_1$. Since the assignment matrix A of $\mathfrak{J}_1$ is in the Jordan canonical form, we present two methods to check the termination of two special classes of linear programs, according to the number of Jordan blocks in A. Namely, we give a simple method to decide the termination of a special class of linear loops whose assignment matrices consist only of one Jordan block with positive real eigenvalue, i.e., $A = J(\lambda), \lambda > 0$. Furthermore, for these special loops, we construct a subset of the set of nonterminating points, which enables us to analyze the termination of this kind of loops only by determining whether the subset is empty or not. This result can also be generalized to determine the termination of another special class of linear programs, whose assignment matrices consist only of finitely many Jordan blocks with the same eigenvalue. Second, for the general program $\mathfrak{J}_1$ whose assignment matrix is $A = diag(J_1(\lambda_1), ..., J_s(\lambda_s))$, $\lambda_i > 0$, a recursive decision process, which reduces the termination of the general class of programs to that of the above mentioned two special classes of programs, is developed to analyze the termination of $\mathfrak{J}_1$ in [17].

In this paper, we will show that for the general Program $\mathfrak{J}_1$ whose assignment matrix is in the Jordan canonical form, i.e., $A = diag(J_1(\lambda_1), ..., J_s(\lambda_s))$, $\lambda_i > 0$, a subset of the set NT of its nonterminating points can still be constructed. It will be shown that such a constructed subset has two properties:

- Program $\mathfrak{J}_1$ is nonterminating over the reals if and only if such a subset is not empty;
- The constructed subset is $A^{\widehat{B}}$-invariant for some positive integer $\widehat{B}$.

The above two properties suggest two methods for checking the termination of Program $\mathfrak{J}_1$, respectively. Clearly, they also suggest two methods for checking the termination of $\mathbf{P}_1$, since the termination of $\mathbf{P}_1$ is equivalent to that of $\mathfrak{J}_1$. By the first property, Checking if Program $\mathbf{P_1}$ terminates is equivalent to checking if the constructed subset is empty. And such a subset can be characterized by semi-algebraic systems. Therefore, for Program $\mathbf{P}_1$, in our first method, we first need to reduce $\mathbf{P}_1$ to $\mathfrak{J}_1$ by computing the Jordan canonical form of the assignment matrix $\widetilde{A}$ of $\mathbf{P}_1$. And then, we construct the desired subset of the set NT of non-terminating points of $\mathfrak{J}_1$ and check whether such a subset is empty or not. Different from the methods given by Tiwari, Braverman and Rebiha et al., any point in such a subset must be a non-terminating point, rather than a N-nonterminating point. In addition, our first method is different from the method given in [17], since the latter is a recursive procedure. Besides, by the second property as above, if the constructed subset is not empty, then there exists an $A^{\widehat{B}}$-invariant set, which can be expressed as a quantified formula over the theory of linear arithmetic interpreted over the reals, in the region specified by the loop conditions of Program $\mathfrak{J}_1$. This suggests the second decision method for the termination of $\mathbf{P}_1$. Also, our second method is different from Theorem 3 in [10], since Theorem 3 given by Tiwari just can deal with the termination of two variables loops. The reason is that Tiwari's Theorem 3 depends on the fact that in 2-dimensional case, the set NT of nonterminating points of $\mathbf{P_1}$ will be an $\widetilde{A}$-invariant sector and it can be specified by its two boundary rays. It has been pointed out by Tiwari that Theorem 3 in [10] can not be generalized to higher dimensions since the region NT in high-dimensional case may not be specified by finitely many hyperplane boundaries. In contrast, our methods do not need to construct NT, but just needs to construct a subset of NT, which can be specified by semi-algebraic systems consisting of finitely many inequalities and equalities.

The rest of the paper is organized as follows. In Section 2, we recall some important results presented in [10]. In Section 3, For Program $\mathfrak{J}_1$, we construct a subset of the set NT of non-terminating points of Program $\mathfrak{J}_1$ and prove that such a subset has the two properties as above. In Section 4, an example is given to illustrate our methods. Finally, we conclude the paper in Section 5.

2. Previous results

In [10], Tiwari establishes the decidability of the termination problem for linear loops of the form $\mathbf{P_1}$. Generally speaking, we say that Program $\mathbf{P_1}$ is nonterminating over the reals, if there is a point $X \in \mathbb{R}^L$, such that $B\widetilde{A}^n X > 0$ holds for all $n \geq 0$. Otherwise, if such a point does not exist, then Program $\mathbf{P_1}$ terminates over the reals. Define $NT = \{X \in \mathbb{R}^L : \widetilde{B}X > 0, \widetilde{B}\widetilde{A}X > 0, \widetilde{B}\widetilde{A}^2 X > 0, \ldots, \widetilde{B}\widetilde{A}^n X > 0, \ldots\}$ to be the set of all points on which Program $\mathbf{P_1}$ does not terminate. Each point in NT is called nonterminating point. At the same time, let $\mathbb{Z}_{\geq 0}$ be a set of nonnegative integers, and denote by NT^e the set of X's for which $A^N X \in NT$ for some nonnegative integer N, i.e.,

$$NT^e = \{X \in \mathbb{R}^L : \exists N \in \mathbb{Z}_{\geq 0}, \widetilde{A}^N X \in NT\}.$$

For convenience, we call every point in NT^e N-nonterminating point. Given a mapping $\mathbb{T} : \mathbb{R}^d \to \mathbb{R}^d$ and a set $\mathbb{S} \subseteq \mathbb{R}^d$, we say $\mathbb{S}$ is $\mathbb{T}$-invariant, if for each element $s \in \mathbb{S}$, $\mathbb{T}(s) \in \mathbb{S}$. According to the definition of invariant sets, it is easy to see that NT and NT^e mentioned above are both $\widetilde{A}$-invariant. Now let us introduce two important results from [10], which can reduce the termination problem of Program $\mathbf{P_1}$ to that of a simper class of programs.

Theorem 1. *([10]) Let P be an invertible matrix. Then the program*

$$\boldsymbol{P_1}\text{:} \quad \textit{while } (\widetilde{B}X > 0) \ \{X := \widetilde{A}X\}$$

is terminating if and only if the program

$$\boldsymbol{P_2}\text{:} \quad \textit{while } (\widetilde{B}PY > 0) \ \{Y := P^{-1}\widetilde{A}PY\}$$

is terminating.

By Theorem 1, the termination problem of Program $\boldsymbol{P}_1$ can be reduced to determining the termination of Program $\boldsymbol{P}_2$. Let $P^{-1}\widetilde{A}P = diag(J_1, \ldots, J_\ell)$, where J_i is the i-th Jordan block in the Jordan canonical form of the assignment matrix $\widetilde{A}$. Let λ_i be the eigenvalue corresponding to the i-th Jordan block J_i. Partition the variables in Y into k segments, say, $Y_1, ..., Y_\ell$, and Program $\boldsymbol{P}_2$ can be rewritten as the form:

$$\begin{aligned} \boldsymbol{P_3}\text{:} \quad & \textit{while}(\widetilde{B}_1 Y_1 + \cdots + \widetilde{B}_\ell Y_\ell > 0) \\ & \{Y_1 := J_1 Y_1; \ldots; Y_\ell := J_\ell Y_\ell\}. \end{aligned}$$

Where $\widetilde{B}_i$'s are obtained by partitioning the matrix $\widetilde{B}P$. Thus, $\mathbf{P}_2 \triangleq \mathbf{P}_3$. Let $\Lambda = \{1, 2, \ldots, \ell\}$ be the set of indices. Define the set $\Lambda_+ = \{i \in \Lambda : \lambda_i > 0\}$. Denote by $|\Lambda_+| = m$ the number of elements in Λ_+. The following theorem presented in [10] shows that we can ignore the state space corresponding to negative and complex eigenvalues.

Theorem 2. *([10]) Let P be an invertible matrix. The program*

$$\textbf{\textit{P}}_3\text{:}\quad while\ (\sum_{j\in\Lambda}\widetilde{B}_jY_j>0)\ \ \{Y_j:=J_jY_j;\ j\in\Lambda\}$$

is terminating if and only if the program

$$\textbf{\textit{P}}_4\text{:}\quad while\ (\sum_{j\in\Lambda_+}\widetilde{B}_jY_j>0)\ \ \{Y_j:=J_jY_j;\ j\in\Lambda_+\}$$

is terminating.

In terms of Theorem 2, we know that if Program $\boldsymbol{P}_4$ does not terminate on input $Y_j := \mathbf{c_j}$ $(j \in \Lambda_+)$, then Program $\boldsymbol{P}_3$ does not terminate on input $Y_j := \mathbf{c_j}(j \in \Lambda_+)$ and $Y_j := 0(j \notin \Lambda_+)$. Conversely, if Program $\boldsymbol{P}_3$ does not terminate, then Program $\boldsymbol{P}_3$ has a non-terminating point Y^* with $Y_j^* = 0(j \notin \Lambda_+)$. Therefore, Theorem 2 tells us that we can reduce the termination of Program $\boldsymbol{P}_1$ to the termination of Program $\boldsymbol{P}_4$. It also shows that we just need to consider the eigenspaces corresponding to positive eigenvalues of $\widetilde{A}$. For convenience, Program $\mathbf{P}_4$ can be rewritten as the following form.

$$\mathfrak{J}_1:\quad while\ (BZ>0)\ \ \{Z:=AZ\}$$

where $B = (\widetilde{B}_j :\ j \in \Lambda_+) \in \mathbb{R}^{s\times r}$, $Z = Y_+ = \bigcup_{j\in\Lambda_+} Y_j = (z_1, ..., z_r)^T \in \mathbb{R}^r$ and

$$A = diag(J_j :\ j \in \Lambda_+) = diag(J_{k_1}, ..., J_{k_m})$$

is a $r \times r$ diagonal matrix and J_{k_i} is a Jordan block with positive real eigenvalue λ_i, for $i = 1, ..., m$. Without loss of generality, assume that $k_i = i$ for $i = 1, ..., m$. That is, assume that $A = diag(J_1, ..., J_m)$. Clearly, $\mathbf{P}_4 \triangleq \mathfrak{J}_1$. Let $\sigma_{\widetilde{A}}$ be a set of all the distinct positive eigenvalues of $\widetilde{A}$. Thus, it is not difficult to see that

$$r = \sum_{\lambda_k\in\sigma_{\widetilde{A}}} mul(\lambda_k). \tag{1}$$

Where $mul(\lambda_k)$ denotes the algebraic multiplicity of λ_k. If $m = 1$, then we see that the assignment matrix A of $\mathfrak{J}_1$ consists only of one Jordan block, i.e., $A = J$.

3. Termination Decision of Program $\mathfrak{J}_1$

In this section, we analyze the termination of the below Program $\mathfrak{J}_1$, which has the following general form.

$$\mathfrak{J}_1 : \quad \textit{while } (BZ > 0) \ \{Z := AZ\}.$$

Where $A = diag(J_1, J_2, ..., J_m)$ with each Jordan block J_i having the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix} \quad (\lambda_i > 0) \tag{2}$$

Given two integers $a, b(a < b)$, define $\overline{[a, b]} = \{a, a+1, ..., b-1, b\}$. Let r_i be the dimension of J_i. Thus, the dimension of A is $r = \sum_{i=1}^{m} r_i$. B is an $s \times r$ real matrix and can be written as $B = (b^1, ..., b^s)^T$, and $b^v = (b^v_1, ..., b^v_m)$, for $v \in \overline{[1, s]}$. The variable vector Z may be partitioned into m subvectors, according to the dimension of each Jordan block J_i, i.e. $Z = (\mathbf{z}_1, ..., \mathbf{z}_m)^T$.

We say Program $\mathfrak{J}_1$ is nonterminating, if there exists at least one point Z^*, such that for any $n \in \mathbb{Z}_{\geq 0}$, we have $Cond(n, Z^*) = BA^n Z^* > 0$. Therefore, to determine the termination of $\mathfrak{J}_1$, we first need to get the expression of $Cond(n, Z)$.

After n iterations, the loop conditions of $\mathfrak{J}_1$ can be written as

$$Cond(n, Z) = \begin{pmatrix} Cond_1(n, Z) \\ Cond_2(n, Z) \\ \vdots \\ Cond_s(n, Z) \end{pmatrix} = \begin{pmatrix} b^1_1 J^n_1 \mathbf{z}_1 + b^1_2 J^n_2 \mathbf{z}_2 + \cdots + b^1_m J^n_m \mathbf{z}_m \\ b^2_1 J^n_1 \mathbf{z}_1 + b^2_2 J^n_2 \mathbf{z}_2 + \cdots + b^2_m J^n_m \mathbf{z}_m \\ \vdots \\ b^s_1 J^n_1 \mathbf{z}_1 + b^s_2 J^n_2 \mathbf{z}_2 + \cdots + b^s_m J^n_m \mathbf{z}_m \end{pmatrix}$$

Clearly, Program $\mathfrak{J}_1$ is nonterminating if and only if there exists a point $Z \in \mathbb{R}^r$, such that $Cond_v(n, Z) > 0$ holds for all $v = 1, 2, ..., s$. And

$$Cond_v(n, Z) = b^v_1 J^n_1 \mathbf{z}_1 + b^v_2 J^n_2 \mathbf{z}_2 + \cdots + b^v_m J^n_m \mathbf{z}_m \tag{3}$$

corresponds exactly to the expression after n iterations of loop condition of Program $\mathfrak{U}_v$,

$$\mathfrak{U}_v : \quad \textit{while } (C^T Z > 0) \ \{Z := AZ\},$$

where $C^T = b^v = (b_1^v, ..., b_m^v)$ and A is the same with the assignment matrix of $\mathfrak{J}_1$. Define

$$NT = \{Z \in \mathbb{R}^r : \ Cond(n, Z) > 0 \ for \ all \ n \in \mathbb{Z}_{\geq 0}\},$$

and

$$NT_v = \{Z \in \mathbb{R}^r : \ Cond_v(n, Z) > 0 \ for \ all \ n \in \mathbb{Z}_{\geq 0}\}.$$

Obviously, $NT = \bigcap_{v=1}^{s} NT_v$. Where NT and NT_v are the sets of non-terminating points of $\mathfrak{J}_1$ and $\mathfrak{U}_v$, respectively. It is easy to see that Program $\mathfrak{J}_1$ is nonterminating if and only if

$$NT = \bigcap_{v=1}^{s} NT_v \neq \emptyset.$$

Therefore, the termination of $\mathfrak{J}_1$ is easily reduced to the termination of $\mathfrak{U}_v$. In the following, we just need to consider the termination of Program $\mathfrak{U}_v$, and the established results for Program $\mathfrak{U}_v$ can be naturally generalized to Program $\mathfrak{J}_1$.

In general, if the set NT_v of non-terminating points of Program $\mathfrak{U}_v$ is constructed, then the termination of $\mathfrak{U}_v$ can be determined by checking whether or not NT_v is empty. However, it has been pointed out that the construction of NT or NT_v is difficult in [10]. In the following, for Program $\mathfrak{U}_v$, we will construct the subset NT_v^o of NT_v, which can be characterized by semi-algebraic systems. And we will claim that

$$NT_v \neq \emptyset \text{ if and only if } NT_v^o \neq \emptyset.$$

Also, we will show that NT_v^o is $A^{\widehat{B}}$-invariant for some positive integer $\widehat{B}$, i.e., there exists some positive integer $\widehat{B}$, such that for any $Z \in NT_v^0$, $A^{\widehat{B}} Z \in NT_v^0$.

We first consider two special forms of Program $\mathfrak{U}_v$ and establish decision methods for the termination of them. And then, two decision approaches will be presented to determine the termination of Program $\mathfrak{U}_v$ having a more general form.

3.1. The Special Case: $A = J$

In the subsection, we consider the termination for the special form $\mathfrak{U}_{v0}$ of $\mathfrak{U}_v$, as follows:

$$\mathfrak{U}_{v0} : \quad while \ (C^T Z > 0) \ \ \{Z := AZ\}$$

Where $C^T = b^v \in \mathbb{R}^{1\times r}, Z \in \mathbb{R}^{r,}$ $A = J_{r\times r} \in \mathbb{R}^{r\times r}$, and J is a Jordan block with positive real eigenvalue λ, in the following form:

$$A_{r\times r} = J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \quad (\lambda > 0) \tag{4}$$

By knowledge in linear algebra, we know that the n-th power of the matrix J can be explicitly written as

$$J^n = A^n_{r\times r} = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \cdots & \binom{n}{r-1}\lambda^{n-(r-1)} \\ 0 & \lambda^n & n\lambda^{n-1} & \cdots & \binom{n}{r-2}\lambda^{n-(r-2)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \cdots & 0 & \lambda^n \end{pmatrix}$$

Where $\lambda > 0$, r is the dimension of the Jordan block J, and $\binom{n}{k} = 0$ for $n < k$. According to Formula (3), the expression after n iterations of the loop conditions of Program $\mathfrak{U}_{v0}$ can be written as

$$Cond_v(n, Z) = C^T A^n Z = b^v J^n Z = \lambda^n f(n, Z),$$

where $f(n, Z) = \sum_{i=0}^{r-1} a_i(Z)n^i$. For example, consider the following linear loop,

$$while(3z_1 + 4z_2 > 0) \;\; \{Z := AZ\},$$

where $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\lambda > 0$. Let $C = (3, 4)^T$. The loop condition of this loop after n-th iteration can be written as the form,

$$\begin{aligned} Cond_v(n, Z) = C^T A^n Z &= (3, 4) \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} (z_1, z_2)^T \\ &= \lambda^n(3\lambda^{-1}z_2 n + 3z_1 + 4z_2) = \lambda^n f(n, Z). \end{aligned}$$

Where $f(n, Z) = a_1(Z)n + a_0(Z) = (3\lambda^{-1}z_2)n + (3z_1 + 4z_2)$. In fact, for Program $\mathfrak{U}_{v0}$, $f(n, Z)$ can be regarded as a polynomial in n of degree $d'_0 \leq r - 1$. If $d'_0 < r - 1$, then let $a_i(Z) \equiv 0$ for all $d'_0 < i \leq r - 1$, and $f(n, Z)$ can be written as $f(n, Z) = \sum_{i=0}^{d'_0} a_i(Z)n^i = \sum_{i=0}^{r-1} a_i(Z)n^i$.

We know that Program $\mathfrak{U}_{v0}$ is nonterminating, if there exists at least one point Z^*, such that for any $n \in \mathbb{Z}_{\geq 0}$, we have $Cond_v(n, Z^*) > 0$. Since $\lambda > 0$, $Cond_v(n, Z^*) > 0$ is equivalent to $f(n, Z^*) > 0$. Therefore, to determine if Program $\mathfrak{U}_{v0}$ is nonterminating, we have to check if there exists at least one point Z^*, such that for any $n \in \mathbb{Z}_{\geq 0}$, $f(n, Z^*) > 0$ holds.

In general, finding the set NT_v of nonterminating points of $\mathfrak{U}_{v0}$ is difficult, since the behavior of a program is very complicated, especially in high-dimensional cases [10]. However, for Programs $\mathfrak{U}_{v0}$, a method is established to construct a subset NT_v^o of NT_v, which can be characterized by semi-algebraic systems. This enables us to get the desired nonterminating points by solving such systems, if the subset is nonempty. The following proposition will play a critical role in the following text. We first state it as follows.

Proposition 1. *Given a polynomial* $h(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,\ a_n \neq 0, h(x) \in \mathbb{R}[x]$. *Let* $\xi(x,\ b) = h(x+b)(b \geq 0)$. *Then, when* $b \to +\infty$, *there exists a real number* $b^* \geq 0$, *such that for any* $b \geq b^*$, *the number of variation of coefficients of* $\xi(x,\ b)$ *is zero.*

Proof. The proof is simple. Set

$$\begin{aligned}\xi(x,b) &= h(x+b)\\ &= a_n(x+b)^n + a_{n-1}(x+b)^{n-1} + \cdots + a_1(x+b) + a_0\\ &= \beta_n(b)x^n + \beta_{n-1}(b)x^{n-1} + \cdots + \beta_1(b)x + \beta_0(b)\end{aligned}$$

Next, we first claim that $lcoeff(\beta_j(b), b) = \binom{n}{n-j}a_n$, for $j = 0, 1, ..., n$. Where, $lcoeff(\diamond, \circ)$ is a function, which returns the leading coefficient of $\diamond$ w.r.t the indeterminate $\circ$. Collecting terms from $\xi(x,b)$ according to the exponents of x, we have

$$\begin{aligned}\xi(x,b) &= a_n x^n + (a_n\binom{n}{1}b + a_{n-1})x^{n-1} + (a_n\binom{n}{2}b^2 +\\ &\quad a_{n-1}\binom{n-1}{1}b + a_{n-2})x^{n-2} + \cdots + (a_n b^n + a_{n-1}b^{n-1}\\ &\quad + \cdots + a_1 b + a_0)\\ &= \sum_{j=0}^{n}\sum_{i=j}^{n}(\binom{i}{i-j}a_i b^{i-j})x^j\end{aligned} \tag{5}$$

Clearly, for all $j = 0, 1, ..., n$, we have $lcoeff(\beta_j(b), b) = \binom{n}{n-j} a_n$. Next, there are two cases to be considered, $a_n > 0, a_n < 0$. We just consider the first case when $a_n > 0$, and the analysis of the second case is similar to that of the first case.

(I) When $a_n > 0$. Since a_n lies in the leading coefficient of $\beta_j(b)$, we have $\beta_j(b) \to +\infty$, as $b \to +\infty$. Thus, for any given $G_j > 0$, there must exist $B_j > 0$, such that when $b > B_j$, we have $\beta_j(b) > G_j > 0$. Set $b^* = \overline{max\{B_0, B_1, ..., B_{n-1}\}} + 1$, when $b \geq B^*$, we have $\beta_j(b) > 0$, for each $j \in \overline{[0, n-1]}$. It immediately follows that when $b \geq B^*$, the number of variation of coefficients of $\xi(x,\ b)$ becomes zero. This completes this proposition. □

According to Proposition 1, we have

$\mathfrak{a}$) if $a_n > 0$, then there must exist $b \in \mathbb{Z}_{\geq 0}$ satisfying the semi-algebraic system $\mathfrak{S}_+ = \{\beta_n(b) > 0, \beta_{n-1}(b) > 0, \ldots, \beta_1(b) > 0, \beta_0(b) > 0, b \geq 0\}$.

$\mathfrak{b}$) if $a_n < 0$, then there must exist $b \in \mathbb{Z}_{\geq 0}$ satisfying the semi-algebraic system $\mathfrak{S}_- = \{\beta_n(b) < 0, \beta_{n-1}(b) < 0, \ldots, \beta_1(b) < 0, \beta_0(b) < 0, b \geq 0\}$.

It is easy to see that any nonzero solution of $\mathfrak{S}_+(or\ \mathfrak{S}_-)$ can be regarded as a bound for positive roots of $h(x)$ by Descartes' rule of signs. According to Proposition 1, we can get the following result.

Theorem 3. *With the above notion. Program $\mathfrak{U}_{v0}$ is non-terminating if and only if there exists $k \in \overline{[0, r-1]}$, such that the following semi-algebraic set*

$$\begin{aligned} S_k = \{ & Z \in \mathbb{R}^r : \ a_v(Z) = 0 \ for\ all\ v > k, \\ & a_v(Z) > 0 \ for\ all\ v \leq k\} \end{aligned}$$

has solutions.

Proof. Suppose that there exists k, such that S_k has at least one solution, say $Z^* \in S_k$. Then for all $n \in \mathbb{Z}_{\geq 0}$, the sign of $f(n, Z^*)$ having degree k w.r.t n remains positive. Hence, Program $\mathfrak{U}_{v0}$ does not terminate on Z^*. For the converse, assume that Program $\mathfrak{U}_{v0}$ does not terminate on $Z^* \in NT$. That is, for any $n \in \mathbb{Z}_{\geq 0}$, $f(n, Z^*) > 0$. Next, we show that there exists a certain $k \in \overline{[0, r-1]}$ such that S_k is nonempty. Without loss of generality, suppose that the degree of $f(n, Z^*)$ is k. So, we have $a_v(Z^*) = 0$ for all $v > k$ and $a_k(Z^*) > 0$. Since $C^T A^{n+b} Z^* = \lambda^{n+b} f(n + b, Z^*)$, by Proposition 1, there exists $b = b^* \in \mathbb{Z}_{\geq 0}$, such that the coefficients of $f(n+b^*, Z^*)$ are all positive. Hence, for all $n \in \mathbb{Z}_{\geq 0}$, we have $f(n + b^*, Z^*) > 0$. Since

$$\begin{aligned} \lambda^{n+b^*} f(n + b^*, Z^*) &= C^T A^{n+b^*} Z^* = C^T A^n (A^{b^*} Z^*) \\ &= \lambda^n f(n, A^{b^*} Z^*), \quad (\lambda > 0), \end{aligned} \tag{6}$$

we know that $f(n, A^{b^*}Z^*) = \lambda^{b^*} f(n+b^*, Z^*)$. By the analysis of the sign of the coefficients of $f(n+b^*, Z^*)$ stated above, it follows that $Z^{**} = A^{b^*}Z^*$ satisfies all of the inequalities in S_k. The remaining task is to claim that Z^{**} also satisfies the equations in S_k. To prove this, substituting $Z = Z^{**}$ into $C^T A^n Z$, we have $\lambda^n f(n, Z^{**}) = \lambda^n f(n, A^{b^*}Z^*) = C^T A^{n+b^*} Z^* = \lambda^{n+b^*} f(n+b^*, Z^*)$, $(\lambda > 0)$. Since $\lambda^n f(n, Z) = \lambda^n \sum_{i=0}^{r-1} a_i(Z) n^i$, we can get that

$$\begin{aligned} \lambda^n f(n, Z^{**}) &= \lambda^n \sum_{i=0}^{r-1} a_i(Z^{**}) n^i \\ &= \lambda^{n+b^*} f(n+b^*, Z^*) = \lambda^{n+b^*} \sum_{i=0}^{r-1} \beta_i(Z^*) n^i. \end{aligned} \tag{7}$$

By Formula (5), it is not difficult to see that

$$\beta_i(Z^*) = c_i a_i(Z^*) + \cdots + c_{r-1} a_{r-1}(Z^*), \tag{8}$$

for $i = 0, ..., r-1$. Where $c_0, ..., c_{r-1} \in \mathbb{R}$. Therefore, since $a_i(Z^*) = 0$ for all $i > k$, we have $\beta_{k+1}(Z^*) = \cdots = \beta_{r-1}(Z^*) = 0$. It immediately follows that $a_{k+1}(Z^{**}) = \cdots = a_{r-1}(Z^{**}) = 0$. Thus, $Z^{**} \in S_k$. This completes the proof of the theorem. □

Remark 1. Note that in Theorem 3, if we replace r by $r_\perp$, then Theorem 3 still holds, since we can write $f(n, Z)$ as $f(n, Z) = \sum_{i=0}^{d'_0} a_i(Z) n^i = \sum_{i=0}^{r_\perp - 1} a_i(Z) n^i$. Where $a_i(Z) \equiv 0$ for all $d'_0 < i \le r_\perp - 1$ and $r_\perp$ is a positive integer number greater than or equal to r. Namely, $r_\perp$ is an upper bound of r. Clearly, the semi-algebraic systems S_k's containing the inequality "$0 > 0$" must be an empty set. In addition, "$0 \equiv 0$" can be deleted directly from the semi-algebraic systems S_k's containing "$0 \equiv 0$", since "$0 \equiv 0$" is always true. Thus, we have

$$\bigcup_{k=0}^{d'_0} S_k = \bigcup_{k=0}^{r-1} S_k = \bigcup_{k=0}^{r_\perp - 1} S_k,$$

since $S_j \equiv \emptyset$ for any $j > d'_0$. Therefore, in practical computation, we do not need to know the precise degree of $f(n, Z)$. In other words, we just need to know the upper bound of the dimension of the assignment matrix A in Program $\mathfrak{U}_{v0}$. By Theorem 3, it is not difficult to see that if S_k is nonempty, then any one solution of S_k is a non-terminating point on which Program $\mathfrak{U}_{v0}$ does not terminate. Thus, $NT_v^o = \bigcup_{k=0}^{r-1} S_k \subseteq NT_v$.

By the arguments in the proof of Theorem 3, it is very easy to get the following corollary.

Corollary 1. *With the above notion. The subset S_k of the set NT_v of non-terminating points of Program $\mathfrak{U}_{v0}$ is A-invariant.*

Proof. Without loss of generality, let

$$f(n, Z) = \sum_{i=0}^{d_0'} a_i(Z)n^i = \sum_{i=0}^{r-1} a_i(Z)n^i,$$

and $a_i(Z) \equiv 0$(zero polynomial) for $i > d_0'$. That is, $f(n, Z)$ is regarded as a polynomial in n and the degree of $f(n, Z)$ w.r.t n is $d_0' \le r - 1$. Take arbitrarily a point Z^* from S_k. Next, we will show that $AZ^* \in S_k$. Because $Z^* \in S_k$, we have $a_i(Z^*) = 0$ for all $i > k$ and $a_i(Z^*) > 0$ for all $i \le k$. Therefore, the degree of $f(n, Z^*)$ is $k \le d_0'$ and all the coefficients of $f(n, Z^*)$ are positive. Obviously, all the coefficients of $f(n + 1, Z^*)$ must be positive. By Equation (6), we have $f(n, AZ^*) = \lambda f(n + 1, Z^*)$. Because $\lambda > 0$, it follows that $a_i(AZ^*) > 0$ for $i \le k$. To prove AZ^* satisfies all the equations in S_k, we adopt the same strategy presented in the proof of Theorem 3. Let $Z^{**} = AZ^*$. By Equation (7) and (8), we get $a_i(AZ^*) = 0$ for $i > k$. This completes the proof of the corollary.

□

Obviously, $NT_v^o = \bigcup_{k=0}^{r-1} S_k$ is A-invariant. By Corollary 1, if $S_k \neq \emptyset$, then there exists a nonempty A-invariant subset of NT_v, charaterized by $a_i(Z) = 0$ for $i > k$ and $a_i(Z) > 0$ for $i \le k$, on which the loop condition of $\mathfrak{U}_{v0}$ always evaluates to true. This can be expressed as a quantified formula over the reals. Note that since $a_i(Z)$ is always linear, in the following, we may replace $a_i(Z)$ by $\boldsymbol{a}_i Z$. Where $\boldsymbol{a}_i$ is a vector composed of the coefficients of $a_i(Z)$.

Theorem 4. *With the above notion. Program $\mathfrak{U}_{v0}$ is non-terminating iff there exists $k \in \overline{[0, r-1]}$, such that the following sentence is true in the the theory of reals*

$$\begin{aligned}\exists \boldsymbol{a}_{r-1} \cdots \exists \boldsymbol{a}_k \cdots \exists \boldsymbol{a}_0 [\exists Z.(\phi_{S_k}(\boldsymbol{a}_{r-1}, ..., \boldsymbol{a}_0, Z) \wedge C^T Z > 0) \wedge \forall Z. \\ (\phi_{S_k}(\boldsymbol{a}_{r-1}, ..., \boldsymbol{a}_0, Z) \wedge C^T Z > 0 \\ \Rightarrow (\phi_{S_k}(\boldsymbol{a}_{r-1}, ..., \boldsymbol{a}_0, AZ) \wedge C^T AZ > 0))]\end{aligned} \tag{9}$$

where ϕ_{S_k} denotes $\bigwedge_{i=k+1}^{r-1} \boldsymbol{a}_i Z = 0 \wedge \bigwedge_{i=0}^{k} \boldsymbol{a}_i Z > 0$.

Proof. Let $\Omega = \{Z \in \mathbb{R}^r : C^T Z > 0\}$. By Theorem 3, if Program $\mathfrak{U}_{v0}$ is non-terminating, then there must exist $k \in \overline{[0, r-1]}$ such that $S_k \neq \emptyset$. And by Corollary 1, S_k is A-invariant subset of NT_v of $\mathfrak{U}_{v0}$. Thus, $S_k = S_k \cap \Omega$, since $NT_v \subseteq \Omega$. Clearly, $S_k \cap \Omega$ can be characterized by Formula (9). Conversely, if there exists $k \in \overline{[0, r-1]}$ such that Formula (9) is true, then there must exist a nonempty A-invariant subset in the region specified by $C^T Z > 0$. This immediately implies that $\mathfrak{U}_{v0}$ is non-terminating. □

Likewise, in Theorem 4, r can be replaced by any upper bound $r_\perp$ of r. In terms of Theorem 3, analogous analysis can be naturally generalized to Program $\mathfrak{J}_0$, which has the condition matrix $B \in \mathbb{R}^{s \times r}$ and the assignment matrix A consisting of one Jordan block with a positive real eigenvalue. Where B is the same with the condition matrix of $\mathfrak{J}_1$ and $A = J_{r \times r}$. That is, $J_0 \triangleq J_0(B, A) \triangleq J_0(B, J)$. Since the number of the loop conditions of Program $\mathfrak{J}_0$ is $s(s > 1)$, after substituting $Z(n) = A^n Z$ into s loop conditions, we get s polynomials $f_j(n, Z)$ of degree $d_j \leq r - 1$ for $j = 1, ..., s$. Without loss of generality, let $f_j(n, Z) = \sum_{i=0}^{d_j} a_{ij}(Z) n^i = \sum_{i=0}^{r-1} a_{ij}(Z) n^i$, and let $a_{ij}(Z) \equiv 0$ for all $d_j < i \leq r - 1$. Since $a_{ij}(Z)$'s are all linear, for convenience, set $a_{ij}(Z) = \boldsymbol{a}_{ij} Z$. Where $\boldsymbol{a}_{ij}$ is a vector composed of the coefficients of $a_{ij}(Z)$. Let

$$\hat{\boldsymbol{a}}_j = (\boldsymbol{a}_{r-1,j}, ..., \boldsymbol{a}_{k_j,j}, ..., \boldsymbol{a}_{0,j})^T,$$

for $j = 1, ..., s$. Let

$$\begin{aligned} S_{k_j}^j = &\{Z \in \mathbb{R}^r : a_{i,j}(Z) = 0 \ for \ all \ i > k_j, \\ &a_{i,j}(Z) > 0 \ for \ all \ i \leq k_j\} \end{aligned}$$

Similarly, construct $r \cdot s$ semi-algebraic sets, say $S_{k_1}^1, S_{k_2}^2, ..., S_{k_s}^s$, for $(k_1, ..., k_s) \in L^s = \overline{[0, r-1]}^s$.

Corollary 2. *With the above notation. Program $\mathfrak{J}_0$ is nonterminating iff there exists an s-tuple $(k_1, k_2, ..., k_s) \in L^s$, such that*

$$S_{k_1}^1 \bigcap S_{k_2}^2 \bigcap \cdots \bigcap S_{k_s}^s \neq \emptyset.$$

Remark 2. It is easy to see that any one solution of $S_{k_1}^1 \bigcap S_{k_2}^2 \bigcap \cdots \bigcap S_{k_s}^s$ is a non-terminating point on which Program $\mathfrak{J}_0$ does not terminate. As stated in Remark 1, when r is replaced by any upper bound $r_\perp$ of r, Corollary 2 still holds.

Corollary 3. *With the above notation. The subset*

$$\bigcup_{(k_1,k_2,...,k_s)\in L^s} S^1_{k_1}\bigcap S^2_{k_2}\bigcap\cdots\bigcap S^s_{k_s} \tag{10}$$

of the set of nonterminating points of Program $\mathfrak{J}_0$ is A-invariant.

By Theorem 4 and Corollary 3, for Program $\mathfrak{J}_0$, a similar result can be obtained as follows.

Theorem 5. *With the above notion. Program $\mathfrak{J}_0$ is non-terminating iff there exists an s-tuple $(k_1, k_2, ..., k_s) \in L^s$, such that the following sentence is true in the theory of reals*

$$\begin{aligned}&\exists\hat{\boldsymbol{a}}_1\cdots\exists\hat{\boldsymbol{a}}_j\cdots\exists\hat{\boldsymbol{a}}_s[\exists Z.\bigwedge_{j=1}^{s}\phi_{S^j_{k_j}}(\hat{\boldsymbol{a}}_j,Z)\wedge BZ>0\wedge\forall Z.\\ &\quad(\bigwedge_{j=1}^{s}\phi_{S^j_{k_j}}(\hat{\boldsymbol{a}}_j,Z)\wedge BZ>0\Rightarrow(\bigwedge_{j=1}^{s}\phi_{S^j_{k_j}}(\hat{\boldsymbol{a}}_j,AZ)\wedge BAZ>0))]\end{aligned} \tag{11}$$

where $\phi_{S^j_{k_j}}$ denotes $\bigwedge_{i=k_j+1}^{r-1}\boldsymbol{a}_{ij}Z=0\wedge\bigwedge_{i=0}^{k_j}\boldsymbol{a}_{ij}Z>0$.

Example 2. Consider the following linear program.

$$\begin{aligned}Q_3:\ &\textit{while}\ (3z_1+4z_2>0\wedge -z_1+z_2>0)\\ &\{z_1:=z_1+z_2;\ z_2:=z_2\}\end{aligned}$$

Since the assignment matrix is a Jordan block with a positive real eigenvalue 1, the termination of Q_3 can be determined by Corollary 2 or Theorem 5. By computation, we get $f_1(n, Z) = 3z_2n + (3z_1 + 4z_2)$ and $f_2(n, Z) = -z_2n + (-z_1 + z_2)$. Let $a_{11}(Z) = \boldsymbol{a}_{11}Z = (0, 3)^T \cdot Z = 3z_2$, $a_{01}(Z) = \boldsymbol{a}_{01}Z = (3, 4)^T \cdot Z = 3z_1 + 4z_2$, $a_{12}(Z) = \boldsymbol{a}_{12}Z = (0, -1)^T \cdot Z = -z_2$ and $a_{02}(Z) = \boldsymbol{a}_{02}Z = (-1, 1)^T \cdot Z = -z_1 + z_2$. Let $\hat{\boldsymbol{a}}_1 = (\boldsymbol{a}_{11}, \boldsymbol{a}_{01})^T$ and $\hat{\boldsymbol{a}}_2 = (\boldsymbol{a}_{12}, \boldsymbol{a}_{02})^T$. Firstly, according to Corollary 2, we construct 4 semi-algebraic systems: $S^1_0 = \{(z_1, z_2) \in \mathbb{R}^2 :\ 3z_2 = 0, 3z_1 + 4z_2 > 0\}$, $S^1_1 = \{(z_1, z_2) \in \mathbb{R}^2 :\ 3z_2 > 0, 3z_1 + 4z_2 > 0\}$, $S^2_0 = \{(z_1, z_2) \in \mathbb{R}^2 :\ -z_2 = 0, -z_1 + z_2 > 0\}$ $S^2_1 = \{(z_1, z_2) \in \mathbb{R}^2 :\ -z_2 > 0, -z_1 + z_2 > 0\}$. Let $L = [0, 1]$. It is easy to see that for any 2-tuple $(k_1, k_2) \in L^2$, $S^1_{k_1}\bigcap S^2_{k_2} = \emptyset$. This implies that Q_3 is terminating. Let

$$\begin{aligned}&\phi_{S^1_1}=\{\boldsymbol{a}_{11}Z>0,\boldsymbol{a}_{01}Z>0\},\phi_{S^1_0}=\{\boldsymbol{a}_{11}Z=0,\boldsymbol{a}_{01}Z>0\},\\ &\phi_{S^2_1}=\{\boldsymbol{a}_{12}Z>0,\boldsymbol{a}_{02}Z>0\},\phi_{S^2_0}=\{\boldsymbol{a}_{12}Z=0,\boldsymbol{a}_{02}Z>0\}.\end{aligned}$$

By Formula (11) in Theorem 5, we can construct 4 quantified formulae. By quantifier elimination technique, we can find that none of the four quantified formulae holds. So, Q_3 must be terminating.

3.2. The General Case: $A = diag(J_1, J_2, ..., J_m)$

In the subsection, we first consider another special case of Program $\mathfrak{U}_v$: all eigenvalues of A are equal. And then, the termination of $\mathfrak{U}_v$ having general form will be analyzed in the rest of the subsection.

3.2.1. Case 1: The eigenvalues of A are all equal

To avoid confusion of symbol, we denote by $\mathfrak{U}_{v1}$ this special program, which has the following form:

$$\mathfrak{U}_{v1}: \quad while\ (C^T Z > 0)\ \ \{Z := AZ\}$$

Where $C^T = b^v = (b_1^v, ...b_m^v) \in \mathbb{R}^{1\times r}, Z \in \mathbb{R}^{r,} A = diag(J_1(\lambda_1), J_2(\lambda_2), ..., J_m(\lambda_m))$ and $\lambda_1 = \cdots = \lambda_m$.

The termination analysis of $\mathfrak{U}_{v1}$ is similar to that of $\mathfrak{U}_{v0}$. According to Formula (3), the expression after n iterations of the loop condition of $\mathfrak{U}_{v1}$ can be written as the following form

$$\begin{aligned} Cond_v(n, Z) &= C^T A^n Z = b_1^v J_1^n \mathbf{z}_1 + b_2^v J_2^n \mathbf{z}_2 + \cdots + b_m^v J_m^n \mathbf{z}_m \\ &= \lambda^n f_{v1}(n, \mathbf{z}_1) + \lambda^n f_{v2}(n, \mathbf{z}_2) + \cdots + \lambda^n f_{vm}(n, \mathbf{z}_m) \\ &= \lambda^n p_v(n, Z) \end{aligned}$$

Let the degree of $p_v(n, Z)$ w.r.t n is $d_v \leq \max_{i=1}^m\{r_i\} - 1$. That is, $p_v(n, Z) = \sum_{i=0}^{d_v} a_{iv}(Z) n^i$. Set $r_0 = \max_{i=1}^m\{r_i\}$ and let $L = [0, ..., r_0 - 1]$. If we set $a_{iv}(Z) \equiv 0$ for all $d_v < i \leq r_0 - 1$, then $p_v(n, Z)$ can be rewritten as

$$p_v(n, Z) = \sum_{i=0}^{d_v} a_{iv}(Z) n^i = \sum_{i=0}^{r_0-1} a_{iv}(Z) n^i.$$

Since $a_i(Z)$'s are all linear, for convenience, set $a_{iv}(Z) = \boldsymbol{a}_{iv} Z$. Where $\boldsymbol{a}_{iv}$ is a vector composed of the coefficients of $a_{iv}(Z)$. Let

$$\hat{\boldsymbol{a}}_v = (\boldsymbol{a}_{r_0-1,v}, ..., \boldsymbol{a}_{k_j,v}, ..., \boldsymbol{a}_{0,v})^T,$$

Obviously, Program $\mathfrak{U}_{v1}$ is nonterminating if and only if there exists one point Z^*, such that $p_v(n, Z^*) > 0$ for all $n \in \mathbb{Z}_{\geq 0}$. Let

$$\begin{aligned} S_k = &\{Z \in \mathbb{R}^r : a_{iv}(Z) = 0\ for\ all\ i > k, \\ &a_{iv}(Z) > 0\ for\ all\ i \leq k\} \end{aligned}$$

As stated in Theorem 3 and corollary 1, similar results are established as follows.

Corollary 4. *With the above notion. Program $\mathfrak{U}_{v1}$ is nonterminating if and only if there exists $k \in L$, such that $S_k \neq \emptyset$.*

Proof. The proof is similar to what we did in Theorem 3 and we will still use the strategy (and notation) of proof of Theorem 3. One direction of the proof is obvious since $S_k \neq \emptyset$ implies that $\mathfrak{U}_{v1}$ is nonterminating. For the converse, assume that there is a point $Z^* = (\mathbf{z}_1^*, \mathbf{z}_2^*, .., \mathbf{z}_m^*)$, on which $\mathfrak{U}_{v1}$ does not terminate. We next claim that there is a $k \in L$ such that S_k is nonempty. Substituting $Z(n) = A^n Z^*$ into the loop condition, we have $Cond_v(n, Z^*) = \lambda^n p_v(n, Z^*) = \lambda^n \sum_{i=0}^{d_v} a_{iv}(Z^*) n^i = \lambda^n \sum_{i=0}^{r_0-1} a_{iv}(Z^*) n^i$, where $a_{iv}(Z) \equiv 0$ for all $d_v < i \leq r_0 - 1$. Denote by $k \in L$ the degree of $p_v(n, Z^*)$. It follows that $a_{i,v}(Z^*) = 0$ for all $i > k$. Since Program $\mathfrak{U}_{v1}$ does not terminate on Z^*, the leading coefficient $a_{k,v}(Z^*)$ of $p_v(n, Z^*)$ must be a positive number. By Proposition 1, we know that there is $b^* \in \mathbb{Z}_{\geq 0}$ such that the number of variation of the coefficients of $p_v(n + b^*, Z^*)$ is zero. Namely, the coefficients of $p_v(n + b^*, Z^*)$ are all positive. Note that

$$C^T A^{n+b^*} Z^* = \lambda^{n+b^*} p_v(n + b^*, Z^*) = \lambda^n p_v(n, A^{b^*} Z^*) \ \ (\lambda > 0). \tag{12}$$

So, there exists $Z^{**} = A^{b^*} Z^*$ such that $a_{i,v}(Z^{**}) > 0 \ for \ all \ i \leq k$. Next, we will show that Z^{**} satisfies the equations $a_{i,v}(Z) = 0$ for all $i > k$. Since $\lambda^n p_v(n, Z) = \lambda^n \sum_{i=0}^{d_v} a_{iv}(Z) n^i$, by Formula (12), we have

$$\begin{aligned} \lambda^n p_v(n, Z^{**}) &= \lambda^n \sum_{i=0}^{d_v} a_{iv}(Z^{**}) n^i \\ &= \lambda^{n+b^*} p_v(n + b^*, Z^*) = \lambda^{n+b^*} \sum_{i=0}^{d_v} \beta_{iv}(Z^*) n^i. \end{aligned}$$

It is not difficult to see that $\beta_{iv}(Z^*) = c_{iv} a_{iv}(Z^*) + \cdots + c_{d_v,v} a_{d_v,v}(Z^*)$, for $i = 0, ..., r_0 - 1$. Where $c_{iv}, ..., c_{d_v,v} \in \mathbb{R}$. So, if $a_{i(i>k),v}(Z^*) = 0$, we have $\beta_{i,v}(Z^*) = 0$, for $i = k + 1, ..., r_0 - 1$. So, we know that $a_{i(i>k),v}(Z^{**}) = 0$. Hence, $Z^{**} \in S_k$. This completes the proof of the corollary. $\square$

Remark 3. As shown in Remark 1, 2, we can replace r_0 in L by $r_\perp$, where $r_\perp$ is an upper bound of r_0. It is easy to see that any one solution of S_k is a non-terminating point on which Program $\mathfrak{U}_{v1}$ does not terminate. Thus, $NT_v^o = \bigcup_{k=0}^{r_0} S_k \subseteq NT_v$. Especially, by the proof of Corollary 4, we know that if there exists Z^* such that $p_v(n, Z^*) \equiv 0$, i.e., all the coefficients of $p_v(n, Z^*)$ are zero, then for any $b \in \mathbb{Z}_{\geq 0}$, we have $p_v(n, A^b Z^*) \equiv 0$.

Using the similar arguments in the proof of Corollary 4, we can obtain the following result.

Corollary 5. *With the above notion. The subset S_k of the set NT_v of non-terminating points of $\mathfrak{U}_{v1}$ is A-invariant.*

Since S_k is A-invariant, $NT_v^o = \bigcup_{k=0}^{r-1} S_k$ is alos A-invariant. Likewise, if $S_k \neq \emptyset$, then there exists a nonempty A-invariant subset of NT_v, characterized by $a_{iv}(Z) = 0$ for $i > k$ and $a_{iv}(Z) > 0$ for $i \leq k$, on which the loop condition always evaluates to true. This Also can be expressed as a quantified formula over the reals.

Theorem 6. *With the above notion. Program $\mathfrak{U}_{v1}$ is non-terminating iff there exists $k \in L$, such that the following sentence is true in the the theory of reals*

$$\begin{aligned}&\exists \hat{\boldsymbol{a}}_v[\exists Z.\phi_{S_k}(\hat{\boldsymbol{a}}_v, Z) \wedge BZ > 0 \wedge \forall Z.\\ &\quad(\phi_{S_k}(\hat{\boldsymbol{a}}_v, Z) \wedge BZ > 0 \Rightarrow (\phi_{S_k}(\hat{\boldsymbol{a}}_v, AZ) \wedge BAZ > 0))]\end{aligned} \tag{13}$$

where ϕ_{S_k} denotes $\bigwedge_{i=k+1}^{r_0-1} \boldsymbol{a}_{iv} Z = 0 \wedge \bigwedge_{i=0}^{k} \boldsymbol{a}_{iv} Z > 0$.

Similar to Corollary 2 and Theorem 5, the above results for the termination of $\mathfrak{U}_{v1}$ can be naturally generalized to the termination of $\mathfrak{J}_1'$, which is specified by the condition matrix $B \in \mathbb{R}^{s\times r}$ and the assignment matrix A consisting of m Jordan blocks with the same eigenvalues. For the restriction of space, we omit the details.

Note that in Theorem 3 in [10], Tiwari gives a quantifier formula to check termination of two variable loops and points out that his result can not be generalized to higher dimensions since the set NT of non-terminating points of a higher-dimensional loop may not be characterized by finitely many hyperplane boundaries. However, Theorem 4, 5 and 6 show that there still exist quantified formulae, which can be used to check termination of some classes of high-dimensional loops. The reason is that we do not need to construct the set NT_v of non-terminating points of Program $\mathfrak{U}_{v0}$(resp. $\mathfrak{U}_{v1}$), but construct the subset NT_v^o of NT_v. Moreover, the constructed subset can be characterized by finitely many inequalities and equalities. More importantly, the constructed subset has a property: Program $\mathfrak{U}_{v0}$(resp. $\mathfrak{U}_{v1}$) is non-terminating iff the constructed subset NT_v^o of NT_v is nonempty, respectively. At the same time, since the constructed subsets NT_v^o of NT_v of Program $\mathfrak{U}_{v0}$(resp. $\mathfrak{U}_{v1}$) is proven to be A-invariant, we can establish the desired quantifier formulae by which the termination of Program $\mathfrak{U}_{v0}$(resp. $\mathfrak{U}_{v1}$) can be checked.

In the following text, we will consider the general Program $\mathfrak{U}_1$ in which not all of the eigenvalues of A are equal. Also, we will show that a subset NT_v^o of the set NT_v of non-terminating points of Program $\mathfrak{U}_v$ can be constructed, and show that the termination of Program $\mathfrak{U}_v$ is non-terminating iff such a subset is nonempty. Furthermore, we will show that the constructed subset NT_v^o of NT_v of Program $\mathfrak{U}_v$ is $A^{\widehat{B}}$-invariant for some positive integer $\widehat{B}$. Thus, if the constructed subset is nonempty, then there must exist nonempty $A^{\widehat{B}}$-invariant set in the region specified by $C^T Z > 0$. This enables us to check the termination of Program $\mathfrak{U}_v$ by checking the satisfiability of quantified formulas, since such an $A^{\widehat{B}}$-invariant set can be characterized by quantified formulas.

3.2.2. Case 2: Not all of the eigenvalues of A are equal

In this subsection, we will consider the termination problem for the general case of Program $\mathfrak{U}_v$: not all of the eigenvalues of the assignment matrix A of $\mathfrak{U}_v$ are equal.

By Formula (3), we know that the expression after n iterations of the loop condition of $\mathfrak{U}_v$ can be written as

$$Cond_v(n, Z) = b_1^v J_1^n \mathbf{z}_1 + b_2^v J_2^n \mathbf{z}_2 + \cdots + b_m^v J_m^n \mathbf{z}_m,$$

where $J_i \in \mathbb{R}^{r_i \times r_i}$. If there exist two Jordan blocks $J_i(\lambda_i)$ and $J_j(\lambda_j)$ such that $\lambda_i = \lambda_j$, then combine $b_i^v J_i^n \mathbf{z}_i, b_j^v J_j^n \mathbf{z}_j$ into single term. Therefore, collecting all "like" terms in $Cond_v(n, Z)$, we can rewrite $Cond_v(n, Z)$ as

$$Cond_v(n, Z) = \lambda_1^n p_{v1}(n, \mathbf{w}_1) + \lambda_2^n p_{v2}(n, \mathbf{w}_2) + \cdots + \lambda_t^n p_{vt}(n, \mathbf{w}_t),$$

where $\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_t$ is a partition of the variable vector Z, i.e. $Z = \mathbf{w}_1 \cup \mathbf{w}_2 \cup ... \cup \mathbf{w}_t$, and $t \leq m$. Without loss of generality, assume $\lambda_1 > \lambda_2 > \cdots > \lambda_t > 0$. Let $\widetilde{p}_{vk}(n, Z) = p_{vk}(n, \mathbf{w}_k)$ for $1 \leq k \leq t$. Since each $p_{vk}(n, \mathbf{w}_k)$ is only related to these Jordan blocks having the same eigenvalue λ_k, it can be written as:

$$\widetilde{p}_{vk}(n, Z) = p_{vk}(n, \mathbf{w}_k) = \lambda_k^{-n} \mathbf{b}_{vk} \mathbf{J}_{vk}^n \mathbf{w}_k. \quad (14)$$

Where $\mathbf{b}_{vk} = (b^v_{k_1}, ..., b^v_{k_{v_k}})$, $\mathbf{J}_{vk} = diag(J_{k_1}(\lambda_k), ..., J_{k_{v_k}}(\lambda_k))$, $\mathbf{w}_k = (\mathbf{z}_{k_1}, ..., \mathbf{z}_{k_{v_k}})^T$ and $\{k_1, ..., k_{v_k}\} \subseteq \{1, 2, ..., m\}$.

Let d_{vk} be the degree of $\widetilde{p}_{vk}(n, Z)$ w.r.t n, for $k = 1, ..., t$. Clearly, $d_{vk} \leq \hat{r}_{vk} = \max_{u=1}^{v_k}\{r_{k_u}\} - 1 < r_\perp$. Where r_{k_u} is the dimension of $J_{k_u}(\lambda_k)$ and $r_\perp > \max_{i=1}^{m} r_i$. Let

$$\begin{aligned}\widetilde{p}_{vk}(n, Z) &= p_{vk}(n, \mathbf{w}_k)\\ &= \sum_{l=0}^{d_{vk}} a_{vkl}(\mathbf{w}_k)n^l = \sum_{l=0}^{d_{vk}} \widetilde{a}_{vkl}(Z)n^l.\end{aligned}$$

If we set $\widetilde{a}_{v,k,l}(Z) \equiv 0$ for all $d_{vk} < l \leq r_\perp$, then $\widetilde{p}_{vk}(n, Z)$ can be rewritten as

$$\widetilde{p}_{vk}(n, Z) = \sum_{l=0}^{d_{vk}} \widetilde{a}_{v,k,l}(Z)n^l = \sum_{l=0}^{r_\perp} \widetilde{a}_{v,k,l}(Z)n^l.$$

It is not difficult to see that $\widetilde{p}_{vk}(n, Z)$ exactly corresponds to the expression of the loop condition after n iterations of a program such as Program $\mathfrak{U}_{v1}$, which is specified by the assignment matrix $\mathbf{J}_{vk}$ and the condition matrix $\mathbf{b}_{vk}$. For convenience, we redefine $Z = (\mathbf{w}_1, ..., \mathbf{w}_t)^T$. Correspondingly, A can be redefined as $A = diag(\mathbf{J}^n_{v1}, ..., \mathbf{J}^n_{vt})$. $Cond_v(n, Z)$ can be further rewritten as:

$$Cond_v(n, Z) = \lambda_1^n \widetilde{p}_{v1}(n, Z) + \lambda_2^n \widetilde{p}_{v2}(n, Z) + \cdots + \lambda_t^n \widetilde{p}_{vt}(n, Z).$$

Similar to the termination analysis of Program $\mathfrak{U}_{v0}$ and $\mathfrak{U}_{v1}$, for the termination of $\mathfrak{U}_v$, our main idea here is to extract semi-algebraic systems consisting of a finite number of equalities and inequalities from $Cond_v(n, Z)$. And we will show that Program $\mathfrak{U}_v$ is non-terminating iff such semi-algebraic systems have solutions. To do this, we first give some necessary notations, which will be used to construct such the semi-algebraic systems. Define

$$\begin{aligned}T_{v,k-1} = \{Z \in \mathbb{R}^r : \ &\widetilde{a}_{v1l}(Z) = 0, l = 0, ..., d_{v1},\\ &\widetilde{a}_{v2l}(Z) = 0, l = 0, ..., d_{v2},\\ &\quad\vdots\\ &\widetilde{a}_{v,k-1,l}(Z) = 0, l = 0, ..., d_{v,k-1}\},\end{aligned}$$

for $1 < k \leq t$. Especially, when $k = 1$, set $T_{v,0} = \mathbb{R}^r$. By the definition of $T_{v,k-1}$, the first $k-1$ polynomials w.r.t n, $\widetilde{p}_{v1}(n, Z), ..., \widetilde{p}_{v,k-1}(n, Z)$ will be zero polynomials on each point in $T_{v,k-1}$.

Proposition 2. *With the above notion. Given Program $\mathfrak{U}_v$. If $Z^* \in T_{v,k-1}$, then for any $b \in \mathbb{Z}_{\geq 0}$, we have $A^b Z^* \in T_{v,k-1}$.*

Proof. Suppose that there exists $Z^* = (\mathbf{w}_1^*, ..., \mathbf{w}_t^*)^T \in T_{v,k-1}$. Then, by the definition of $T_{v,k-1}$, it follows that for each $j \in \overline{[1, k-1]}$, $\widetilde{a}_{vjl}(Z^*) = a_{vjl}(\mathbf{w}_j^*) = 0$ for all $l = 0, 1, ..., d_{vj}$. That is, for each $j \in \overline{[1, k-1]}$,

$$\widetilde{p}_{vj}(n, Z^*) = p_{vj}(n, \mathbf{w}_j^*) = \lambda_j^{-n} \mathbf{b}_{vj} \mathbf{J}_{vj}^n \mathbf{w}_j^* \equiv 0.$$

Set $Z^{**} = A^b Z^* = (\mathbf{J}_{v1}^b \mathbf{w}_1^*, ..., \mathbf{J}_{vt}^b \mathbf{w}_t^*)^T$. Since $a_{vjl}(\mathbf{w}_j^*) = 0$ for all $l = 0, 1, ..., d_{vj}$, by Remark 3, we get that for any $b \in \mathbb{Z}_{\geq 0}$, $\widetilde{a}_{vjl}(Z^{**}) = a_{vjl}(\mathbf{J}_{vj}^b \mathbf{w}_j^*) = 0$ for all $l = 0, 1, ..., d_{vj}$. Hence, $\widetilde{p}_{vj}(n, Z^{**}) = p_{vj}(n, \mathbf{J}_{vj}^b \mathbf{w}_j^*) \equiv 0$, for all $1 \leq j \leq k-1$. Therefore, $Z^{**} = A^b X^* \in T_{v,k-1}$. □

Given two nonzero polynomials $f(n), g(n) \in R[n]$, let

$$f(n) = f_l n^l + f_{l-1} n^{l-1} + \cdots + f_1 n + f_0,$$
$$g(n) = g_l n^l + g_{l-1} n^{l-1} + \cdots + g_1 n + g_0,$$

where f_i's and g_i's are the coefficients of $f(n), g(n)$ respectively. We say $f(n) \succ_{coe} g(n)$, if $f_i > g_i$ for all $i = 0, 1, ..., l$. Especially, $f(n) \succ_{coe} 0$ denotes that all the coefficients of $f(n)$ are positive. Also, given two functions $F(n), G(n)$, we say $F(n) \succ_{\mathbb{Z}_{\geq 0}} G(n)$(resp.$F(n) \succeq_{\mathbb{Z}_{\geq 0}} G(n)$), if $F(n) > G(n)$ (resp. $F(n) \geq G(n)$) holds for all $n \in \mathbb{Z}_{\geq 0}$.

Proposition 3. *Given two polynomials $f(n), g(n)$ as above. Suppose $f(n) \succ_{coe} 0$, $g(n) \succ_{coe} 0$, and $f(n) \succ_{coe} g(n)$. Then, we have*

(1) for any $h(n) \in R[n]$, if $h(n) \succ_{coe} 0$, then $f(n) \cdot h(n) \succ_{coe} g(n) \cdot h(n)$;

(2) for any $h(n) \in R[n]$, if $g(n) + h(n) \succ_{coe} 0$, then $f(n) + h(n) \succ_{coe} 0$.

(3) $f(n) \succ_{\mathbb{Z} \geq 0} g(n) \succ_{\mathbb{Z} \geq 0} 0$.

Proposition 4. *Given a polynomial*

$$s(n) = s_m n^m + s_{m-1} n^{m-1} + \cdots + s_1 n + s_0,$$

$s_j > 0$ for all $j = 0, ..., m$. Let δ be a positive number greater than 1. Then, there exists a positive number B^*, such that for any $b \geq B^*$ we have

$$s(n) \cdot \delta^b \succ_{coe} s(n+b) \succ_{coe} 0.$$

Proof. The proof is very simple. Let

$$\begin{aligned}&s(n+b)\\&=\beta_m(b)n^m+\beta_{m-1}(b)n^{m-1}+\cdots+\beta_1(b)n+\beta_0(b)\\&=\sum_{j=0}^{m}(\sum_{i=j}^{m}(\binom{i}{i-j}s_ib^{i-j}))\cdot n^j\\&=\sum_{j=0}^{m}\beta_j(b)n^j.\end{aligned}$$

Let $Lcoeff(\beta_j(b),b)=\binom{m}{m-j}s_m$. Clearly, $\beta_j(b)\to+\infty$ as $b\to+\infty$. Let $s(n)\cdot\delta^b=\sum_{j=0}^{m}(s_j\delta^b)n^j$. Since $\delta>1$ implies that $s_j\delta^b\gg\beta_j(b)$ as $b\to+\infty$, there must exist B_1^* such that $s(n)\cdot\delta^b\succ_{coe}s(n+b)$ for any $b\geq B_1^*$. Also, by Proposition 1, we know that there exists $B_2^*\in\mathbb{Z}_{\geq0}$ such that for any $b\geq B_2^*$, $s(n+b)\succ_{coe}0$. Take $B^*=\max\{B_1^*,B_2^*\}$. The proposition is now proved. □

Define

$$s_k(n)=\sum_{j=0}^{\mu_k}s_{kj}n^j \tag{15}$$

be a lower-bound for the exponential expression $(\frac{\lambda_k}{\lambda_{k+1}})^n$, i.e.,

$$(\frac{\lambda_k}{\lambda_{k+1}})^n\succeq_{\mathbb{Z}_{\geq0}}s_k(n)\succ_{\mathbb{Z}_{\geq0}}0, \tag{16}$$

satisfying the following two conditions:

- all its coefficients s_{kj}'s are positive;
- $\mu_k\geq\max_{i=1}^{m}r_i$;

for any $k\in\overline{[1,t-1]}$.

By Proposition 4, given a positive $\delta>1$, the first condition ensures that there must exist a positive integer B^*, such that for any $b\geq B^*$, we have

$$s_k(n)\cdot\delta^b\succ_{coe}s_k(n+b)\succ_{coe}0$$

since all the coefficients of $s_k(n)$ are positive. The second condition ensures that the polynomials $\widetilde{H}_{v,k}(n,Z)$, $\widetilde{H}_{v,k,u}(n,Z)$ defined as below have the same leading term.

The following proposition tells us that such the lower-bound $s_k(n)$ of $(\frac{\lambda_k}{\lambda_{k+1}})^n$ can always be constructed.

Proposition 5. *With the above notion. Given* $\frac{\lambda_k}{\lambda_{k+1}}$ *and any fixed positive integer* μ_k*. Let* $\mathcal{M}_k(n) = \sum_{i=0}^{\mu_k} n^i$*. Then, there must exist* $b^* \geq 0$*, such that*

$$s_k(n) = \frac{1}{\mathcal{M}_k(b^*)}\mathcal{M}_k(n) \tag{17}$$

is a lower bound of $(\frac{\lambda_k}{\lambda_{k+1}})^n$.

Proof. Let $\frac{\lambda_k}{\lambda_{k+1}} = 1 + \epsilon_k (\epsilon_k > 0)$ and $\mathcal{H}_k(n) = \sum_{j=0}^{\mu_k+1} \binom{n}{j} \epsilon_k^j$. Clearly,

$$(\frac{\lambda_k}{\lambda_{k+1}})^n \succeq_{\mathbb{Z}_{\geq 0}} \mathcal{H}_k(n) \succeq_{\mathbb{Z}_{\geq 0}} 1.$$

It is not difficult to see that $\mathcal{H}_k(n)$ is a polynomial in n of degree μ_k+1 and its the leading coefficient is positive. Let $\widetilde{\mathcal{H}}_k(n) = \mathcal{H}_k(n) - \mathcal{M}_k(n)$. Obviously, $\widetilde{H}_k(n)$ is a polynomial of degree $\mu_k + 1$ and its the leading coefficient is positive, since the degree of $\mathcal{H}_k(n)$ is greater than that of $\mathcal{M}_k(n)$. Thus, by Proposition 1, there must exist $b^* > 0$, such that all the coefficients of $\widetilde{H}_k(n + b^*)$ are positive. That is, $\widetilde{H}_k(n + b^*) \succ_{coe} 0$. This implies that $\widetilde{H}_k(n + b^*) \succeq_{Z \geq 0} 0$. Hence, we have

$$\mathcal{H}_k(n) \geq \mathcal{M}_k(n), \tag{18}$$

for all $n \geq b^*$. Clearly, $\mathcal{M}_k(\nu) > 0$ for all $\nu = 0, ..., b^*$. Let $c = \max_{\nu=0}^{b^*} \mathcal{M}_k(\nu) = \mathcal{M}_k(b^*)$. Clearly, $c > 1$. So, we have

$$\mathcal{M}_k(n) \geq \frac{1}{c}\mathcal{M}_k(n) \tag{19}$$

for all $n \in \mathbb{Z}_{\geq 0}$. In addition, we have

$$\mathcal{H}_k(n) \geq \frac{1}{c}\mathcal{M}_k(n) \tag{20}$$

for all $n = 0, ..., b^*$, since $\mathcal{H}_k(n) \succeq_{Z \geq 0} 1$ and $\frac{1}{c}\mathcal{M}_k(n) \leq 1$ for all $n = 0, ..., b^*$.

By (18), (19) and (20), it follows that

$$\mathcal{H}_k(n) \geq \frac{1}{c}\mathcal{M}_k(n)$$

for all $n \in \mathbb{Z}_{\geq 0}$. That is,

$$\mathcal{H}_k(n) \succeq_{Z \geq 0} \frac{1}{c}\mathcal{M}_k(n).$$

Because $(\frac{\lambda_k}{\lambda_{k+1}})^n \succeq_{\mathbb{Z}_{\geq 0}} \mathcal{H}_k(n)$, we get

$$(\frac{\lambda_k}{\lambda_{k+1}})^n \succeq_{\mathbb{Z}_{\geq 0}} \frac{1}{c}\mathcal{M}_k(n).$$

Let $s_k(n) = \frac{1}{c}\mathcal{M}_k(n)$. This completes the proof of the proposition. □

By Proposition 5, such a lower bound of $(\frac{\lambda_k}{\lambda_{k+1}})^n$, which satisfies the two conditions mentioned above, can always be obtained. Therefore, we can construct a polynomial expression from the exponential expression $Cond_v(n, Z)$ by replacing $(\frac{\lambda_k}{\lambda_{k+1}})^n$ by its lower bound $s_k(n)$. For example, consider the case when $t = 3$,

$$\begin{aligned} Cond_v(n, Z) &= \lambda_1^n \widetilde{p}_{v1}(n, Z) + \lambda_2^n \widetilde{p}_{v2}(n, Z) + \lambda_3^n \widetilde{p}_{v3}(n, Z) \\ &= \lambda_3^n ((\frac{\lambda_2}{\lambda_3})^n ((\frac{\lambda_1}{\lambda_2})^n \widetilde{p}_{v1}(n, Z) + \widetilde{p}_{v2}(n, Z)) + \widetilde{p}_{v3}(n, Z)). \end{aligned}$$

Replacing $(\frac{\lambda_1}{\lambda_2})^n$ by $s_1(n)$, we construct a polynomial expression

$$s_1(n)\widetilde{p}_{v1}(n, Z) + \widetilde{p}_{v2}(n, Z) + \widetilde{p}_{v3}(n, Z).$$

For convenience, we call the constructed polynomial expression the *polynomial part* in $Cond_v(n, Z)$. In the following, we will show how to extract a semi-algebraic system from the *polynomial part* in $Cond_v(n, Z)$, to characterize the subset NT_v^o of NT_v. Using $s_k(n)$ defined as above, we define $\widetilde{H}_{v,k}(n, Z)$ and $\widetilde{H}_{v,k,u}(n, Z)$ as follows. Let

$$\begin{aligned} \widetilde{H}_{v,k}(n, Z) &= H_{v,k}(n, \mathbf{w}_k) \\ &= s_k(n) p_{vk}(n, \mathbf{w}_k) = s_k(n)\widetilde{p}_{vk}(n, Z) \\ &= \sum_{l=0}^{\mu_k + d_{vk}} (\sum_{i+j=l} s_{ki} a_{vkj}(\mathbf{w}_k)) n^l \\ &= \sum_{l=0}^{\mu_k + d_{vk}} (\sum_{i+j=l} s_{ki} \widetilde{a}_{vkj}(Z)) n^l \\ &= \sum_{l=0}^{\mu_k + d_{vk}} \beta_{vkl}(\mathbf{w}_k) n^l = \sum_{l=0}^{\mu_k + d_{vk}} \widetilde{\beta}_{vkl}(Z) n^l \\ &= \sum_{l \geq \mu_k} \widetilde{\beta}_{vkl}(Z) n^l + \sum_{l < \mu_k} \widetilde{\beta}_{vkl}(Z) n^l. \end{aligned}$$

And let

$$
\begin{aligned}
\widetilde{H}_{v,k,u}(n,Z) &= H_{v,k,u}(n,\mathbf{w}_k,...,\mathbf{w}_u)\\
&= H_{v,k}(n,\mathbf{w}_k) + p_{v,k+1}(n,\mathbf{w}_{k+1}) + \cdots + p_{vu}(n,\mathbf{w}_u)\\
&= s_k(n)p_{vk}(n,\mathbf{w}_k) + p_{v,k+1}(n,\mathbf{w}_{k+1}) + \cdots + p_{vu}(n,\mathbf{w}_u)\\
&= s_k(n)\widetilde{p}_{vk}(n,Z) + \widetilde{p}_{v,k+1}(n,Z) + \cdots + \widetilde{p}_{vu}(n,Z)\\
&= \sum_{l\ge\mu_k}\widetilde{\beta}_{vkl}(Z)n^l + (\sum_{l<\mu_k}\widetilde{\beta}_{vkl}(Z)n^l + \widetilde{p}_{v,k+1}(n,Z)\\
&\qquad\qquad + \cdots + \widetilde{p}_{vu}(n,Z))\\
&= \sum_{l=0}^{\mu_k+d_{vk}}\widetilde{\beta}_{v,k,u,l}(Z)n^l,
\end{aligned}
$$

where $k+1 \le u \le t$. Obviously, $\widetilde{\beta}_{v,k,u,l}(Z) \equiv \widetilde{\beta}_{vkl}(Z)$ for all $l \ge \mu_k$, since the degree of $\widetilde{p}_{v,j}(n,Z)$ w.r.t n is less than μ_k, according to the definition of $s_k(n)$, for $k \le j \le t$.

Proposition 6. *With the above notion. Given a point $Z^* \in \mathbb{R}^r$. If there exists $q_{vk} \in \overline{[0,d_{vk}]}$ such that $\widetilde{a}_{v,k,q_{vk}}(Z^*) > 0$ and $\widetilde{a}_{v,k,j}(Z^*) = 0$ for all $q_{vk} < j \le d_{vk}$, then we have*

(1) $\widetilde{\beta}_{v,k,\mu_k+q_{vk}}(Z^*) > 0$,

(2) $\widetilde{\beta}_{v,k,l}(Z^*) = 0$ for all $\mu_k + q_{vk} < l \le \mu_k + d_{vk}$

Proof. First, when $\mu_k + q_{vk} \le l \le \mu_k + d_{vk}$, we have

$$
\begin{aligned}
\widetilde{\beta}_{vkl}(Z) &= \sum_{i+j=l} s_{ki}\widetilde{a}_{v,k,j}(Z)\\
&= s_{k,\mu_k}\widetilde{a}_{v,k,l-\mu_k}(Z) + s_{k,\mu_k-1}\widetilde{a}_{v,k,l-(\mu_k-1)}(Z)\\
&+ \cdots + s_{k,l-d_{vk}+1}\widetilde{a}_{v,k,d_{vk}-1}(Z) + s_{k,l-d_{vk}}\widetilde{a}_{v,k,d_{vk}}(Z).
\end{aligned}
\tag{21}
$$

(A) When $\mu_k + q_{vk} < l \le \mu_k + d_{vk}$, we have $q_{vk} < l - \mu_k \le d_{vk}$. Then, by hypothesis, since $\widetilde{a}_{v,k,j}(Z^*) = 0$ for all $q_{vk} < j \le d_{vk}$, it follows that $\widetilde{\beta}_{v,k,l}(Z^*) = 0$ for all $d + q_{vk} < l \le \mu_k + d_{vk}$.

(B) When $l = \mu_k + q_{vk}$, we have $l - \mu_k = q_{vk}$. By hypothesis, it follows that $\widetilde{\beta}_{v,k,\mu_k+q_{vk}}(Z^*) = s_{k,\mu_k}\widetilde{a}_{v,k,q_{vk}}(Z^*) > 0$, since $s_{k,\mu_k} > 0$. This completes the proof of the theorem. □

In terms of Propositon 6, since $\widetilde{H}_{vk}(n,Z) = s_k(n)\widetilde{p}_{vk}(n,Z)$, we know that $\widetilde{\beta}_{v,k,l}(Z^*) = 0$ for all $\mu_k + q_{vk} < l \le \mu_k + d_{vk}$ and $\widetilde{\beta}_{v,k,\mu_k+q_{vk}}(Z^*) > 0$. That is, $\widetilde{\beta}_{v,k,\mu_k+q_{vk}}(Z^*)$ is the leading coefficient of $\widetilde{H}_{vk}(n,Z^*)$. In addition, because $\mu_k + q_{vk} \ge \mu_k$, $\widetilde{\beta}_{v,k,\mu_k+q_{vk}}(Z^*) = \widetilde{\beta}_{v,k,u,\mu_k+q_{vk}}(Z^*)$ is also the leading coefficient of $\widetilde{H}_{v,k,u}(n,Z^*)$ for all $k+1 \le u \le t$.

Next, for each $q_{vk} \in \overline{[0,d_{vk}]}$, set

$$S_{\widetilde{p}_{vk,q_{vk}}} = \{Z \in \mathbb{R}^r : \widetilde{a}_{v,k,j}(Z) = 0, for\ all\ j > q_{vk} \\ \widetilde{a}_{v,k,j}(Z) > 0, for\ all\ j \le q_{vk}\}$$

and

$$S_{\widetilde{H}_{v,k,u}} = \{Z \in \mathbb{R}^r :\ \widetilde{\beta}_{v,k,u,l}(Z) > 0,\ for\ all\ l < \mu_k + q_{vk}\}.$$

Then, let

$$\begin{aligned} NT_v^o &= \bigcup_{k=1}^{t} \bigcup_{q_{vk}=0}^{d_{vk}} (T_{v,k-1} \bigcap S_{\widetilde{p}_{vk,q_{vk}}} \bigcap (\bigcap_{u=k+1}^{t} S_{\widetilde{H}_{v,k,u}})) \\ &= \bigcup_{k=1}^{t} \bigcup_{q_{vk}=0}^{d_{vk}} NT^o_{v,k,q_{vk}} \end{aligned} \tag{22}$$

Let $r_\perp$ be any integer greater than d_{vk}'s. Note that NT_v^o can also be rewritten as

$$\begin{aligned} NT_v^o &= \bigcup_{k=1}^{t} \bigcup_{q_{vk}=0}^{r_\perp} (T_{v,k-1} \bigcap S_{\widetilde{p}_{vk,q_{vk}}} \bigcap (\bigcap_{u=k+1}^{t} S_{\widetilde{H}_{v,k,u}})) \\ &= \bigcup_{k=1}^{t} \bigcup_{q_{vk}=0}^{r_\perp} NT^o_{v,k,q_{vk}}, \end{aligned} \tag{23}$$

since $\widetilde{a}_{vkj}(Z) \equiv 0$ for all $j > d_{vk}$. Therefore, $NT^o_{v,k,q_{vk}} = \emptyset$ for all $d_{vk} < q_{vk} \le r_\perp$, since "$0 > 0$" always lies in $NT^o_{v,k,q_{vk}}$ for all $d_{vk} < q_{vk} \le r_\perp$.

Especially, for completeness, let $T_{v,0} = \mathbb{R}^r$ and

$$\bigcap_{u=t+1}^{t} S_{\widetilde{H}_{v,k,u}} = \mathbb{R}^r.$$

Theorem 7. *With the above notion. If $NT_v \ne \emptyset$, then $NT_v^o \ne \emptyset$.*

Proof. If $NT_v \neq \emptyset$, then take arbitrarily $Z^* = (\mathbf{w}_1^*, ..., \mathbf{w}_t^*)^T$ from NT_v. Since Z^* is a nonterminating point, it follows that $Cond_v(n, Z^*) > 0$ holds for all $n \in \mathbb{Z}_{\geq 0}$. Without loss of generality, assume that $\widetilde{p}_{v1}(n, Z^*) = \cdots = \widetilde{p}_{v,k-1}(n, Z^*) \equiv 0$ and $\widetilde{p}_{v,k}(n, Z^*) \not\equiv 0$. That is,

$$\begin{aligned} Cond_v(n, Z^*) = {} & \lambda_1^n \cdot 0 + \lambda_2^n \cdot 0 + \cdots + \lambda_{k-1}^n \cdot 0 \\ & + \lambda_k^n \widetilde{p}_{vk}(n, Z^*) + \cdots + \lambda_t^n \widetilde{p}_{vt}(n, Z^*) \succ_{\mathbb{Z}_{\geq 0}} 0. \end{aligned} \tag{24}$$

Thus, $Z^* \in T_{v,k-1}$. Hence, by Proposition 2, for any $b \in \mathbb{Z}_{\geq 0}$, $A^b X^* \in T_{v,k-1}$. Hence, Equation (24) can be rewritten as the following form,

$$\begin{aligned} Cond_v(n, Z^*) &= \lambda_k^n \widetilde{p}_{vk}(n, Z^*) + \cdots + \lambda_t^n \widetilde{p}_{vt}(n, Z^*) \\ &= \lambda_t^n ((\frac{\lambda_{t-1}}{\lambda_t})^n (\cdots ((\frac{\lambda_{k+1}}{\lambda_{k+2}})^n \cdot ((\frac{\lambda_k}{\lambda_{k+1}})^n \\ & \widetilde{p}_{v,k}(n, Z^*) + \widetilde{p}_{v,k+1}(n, Z^*)) + \widetilde{p}_{v,k+2}(n, Z^*)) + \\ & \cdots)) + \widetilde{p}_{v,t-1}(n, Z^*)) + \widetilde{p}_{v,t}(n, Z^*)) \succ_{\mathbb{Z} \geq 0} 0. \end{aligned} \tag{25}$$

Since $\widetilde{p}_{v,k}(n, Z^*) \not\equiv 0$, without loss of generality, assume that there exists $q_{vk} \in \overline{[0, d_{vk}]}$ such that

$$\widetilde{a}_{v,k,d_{vk}}(Z^*) = \widetilde{a}_{v,k,d_{vk}-1}(Z^*) = \cdots = \widetilde{a}_{v,k,q_{vk}+1}(Z^*) = 0,$$

and

$$\widetilde{a}_{v,k,q_{vk}}(Z^*) \neq 0.$$

That is, $\widetilde{p}_{v,k}(n, Z^*)$ is a polynomial in n of degree q_{vk}, which exactly corresponds to the expression of the loop condition after n iterations of a program such as Program $\mathfrak{U}_{v1}$. Since $Cond_v(n, Z^*) \succ_{\mathbb{Z}_{\geq 0}} 0$, we get $\widetilde{a}_{v,k,q_{vk}}(Z^*) = a_{v,k,q_{vk}}(\mathbf{w}_k^*) > 0$. Hence, By the arguments in the proof of Corollary 4, we know that there must exist a positive integer $b_{q_{vk}}$ such that for any $b \geq b_{q_{vk}}$, we have $A^b Z^* \in S_{\widetilde{p}_{vk,q_{vk}}}$.

Next, we will claim that there exists a integer $B'_{vk} \in \mathbb{Z}_{\geq 0}$, such that for any $b \geq B'_{vk}$, we have $A^b Z^* \in S_{\widetilde{H}_{v,k,u}}$. To prove this, we just need to claim that all the coefficients of $\widetilde{H}_{v,k,u}(n, A^b Z^*)$ are positive for any $b \geq B'_{vk}$.

First, because $\widetilde{H}_{vk}(n, Z) = s_k(n) \cdot \widetilde{p}_{vk}(n, Z)$, by Proposition 6, we get $\widetilde{\beta}_{v,k,l}(Z^*) = 0$ for all $\mu_k + q_{vk} < l \leq \mu_k + d_{vk}$ and $\widetilde{\beta}_{v,k,\mu_k+q_{vk}}(Z^*) > 0$. Where $\widetilde{\beta}_{v,k,\mu_k+q_{vk}}(Z^*)$ is the leading coefficient of $\widetilde{H}_{vk}(n, Z^*)$. Since $\mu_k + q_{vk} \geq \mu_k$, $\widetilde{\beta}_{v,k,\mu_k+q_{vk}}(Z^*) = \widetilde{\beta}_{v,k,u,\mu_k+q_{vk}}(Z^*)$ is the leading coefficient of $\widetilde{H}_{v,k,u}(n, Z^*)$

too, for all $k+1 \le u \le t$. That is to say, $\widetilde{H}_{v,k,u}(n, Z^*)$ is a polynomial in n of degree $\mu_k + q_{vk}$. By Proposition 1, there must exist a positive integer $b_{\widetilde{H}_{vku}}$, such that for any $b \ge b_{\widetilde{H}_{vku}}$, all the coefficients of $\widetilde{H}_{v,k,u}(n+b, Z^*)$ are positive for all $k+1 \le u \le t$, since its leading coefficient $\widetilde{\beta}_{v,k,u,\mu_k+q_{vk}}(Z^*)$ is positive. That is, for any $b \ge b_{\widetilde{H}_{vku}}$,

$$\widetilde{H}_{v,k,u}(n+b, Z^*) \succ_{coe} 0$$

for all $k+1 \le u \le t$. Let

$$B^o_{vk} = \max(\max_{u=k+1}^{t}\{b_{\widetilde{H}_{vku}}\}, b_{q_{vk}}).$$

By Proposition 4, there is a integer $B^*_{vk} \in \mathbb{Z}_{\ge 0}$ such that for any $b \ge B^*_{vk}$, we have

$$s_k(n)(\frac{\lambda_k}{\lambda_{k+1}})^b \succ_{coe} s_k(n+b) \succ_{coe} 0, \tag{26}$$

since $\frac{\lambda_k}{\lambda_{k+1}} > 1$. Take $B'_{vk} = \max(B^o_{vk}, B^*_{vk})$. Hence, for any $b \ge B'_{vk}$, by Proposition 3, we have

$$s_k(n)(\frac{\lambda_k}{\lambda_{k+1}})^b \cdot \widetilde{p}_{vk}(n+b, Z^*) \succ_{coe} s_k(n+b) \cdot \widetilde{p}_{vk}(n+b, Z^*),$$

since $\widetilde{p}_{vk}(n+b, Z^*) \succ_{coe} 0$. Since for any $b \ge B'_{vk}$,

$$\begin{aligned} &\widetilde{H}_{v,k,k+1}(n+b, Z^*) \\ &= s_k(n+b) \cdot \widetilde{p}_{vk}(n+b, Z^*) + \widetilde{p}_{v,k+1}(n+b, Z^*) \succ_{coe} 0, \end{aligned} \tag{27}$$

it directly follows that

$$\begin{aligned} &s_k(n)(\frac{\lambda_k}{\lambda_{k+1}})^b \cdot \widetilde{p}_{vk}(n+b, Z^*) + \widetilde{p}_{v,k+1}(n+b, Z^*) \\ &\succ_{coe} s_k(n+b) \cdot \widetilde{p}_{vk}(n+b, Z^*) + \widetilde{p}_{v,k+1}(n+b, Z^*) \\ &\succ_{coe} 0 \end{aligned} \tag{28}$$

by Proposition 3. Multiplying both sides of Formula (28) by λ^b_{k+1}, we have

$$s_k(n)\lambda_k^b \widetilde{p}_{vk}(n+b, Z^*) + \lambda^b_{k+1}\widetilde{p}_{vk}(n+b, Z^*) \succ_{coe} 0. \tag{29}$$

By Equation (12), we know that $\lambda_k^b \widetilde{p}_{vk}(n+b, Z^*) = \widetilde{p}_{vk}(n, A^b Z^*)$ and $\lambda_{k+1}^b \widetilde{p}_{v,k+1}(n+b, Z^*) = \widetilde{p}_{v,k+1}(n, A^b Z^*)$. Hence, we get

$$s_k(n)\widetilde{p}_{vk}(n, A^b Z^*) + \widetilde{p}_{v,k+1}(n, A^b Z^*) \succ_{coe} 0$$

for any $b \geq B'_{vk}$. This implies that $A^b Z^* \in S_{\widetilde{H}_{v,k,k+1}}$, when $b \geq B'_{vk}$. Because $(\frac{\lambda_{k+1}}{\lambda_{k+2}}) > 1$, by Formula (28), for any $b \geq B'_{vk}$, we have

$$\begin{aligned} &(\frac{\lambda_{k+1}}{\lambda_{k+2}})^b [s_k(n)(\frac{\lambda_k}{\lambda_{k+1}})^b \widetilde{p}_{vk}(n+b, Z^*) + \widetilde{p}_{v,k+1}(n+b, Z^*)] \\ &\succ_{coe} 1 \cdot [s_k(n)(\frac{\lambda_k}{\lambda_{k+1}})^b \widetilde{p}_{vk}(n+b, Z^*) + \widetilde{p}_{v,k+1}(n+b, Z^*)] \\ &\succ_{coe} s_k(n+b)\widetilde{p}_{vk}(n+b, Z^*) + \widetilde{p}_{v,k+1}(n+b, Z^*) \\ &\succ_{coe} 0. \end{aligned} \tag{30}$$

Since for any $b \geq B'_{vk}$,

$$\begin{aligned} &\widetilde{H}_{v,k,k+2}(n+b, Z^*) \\ &= s_k(n+b) \cdot \widetilde{p}_{vk}(n+b, Z^*) + \widetilde{p}_{v,k+1}(n+b, Z^*) \\ &\ + \widetilde{p}_{v,k+2}(n+b, Z^*) \succ_{coe} 0, \end{aligned} \tag{31}$$

by Formula (30) and Proposition 3, we know that

$$\begin{aligned} &(\frac{\lambda_{k+1}}{\lambda_{k+2}})^b [s_k(n)(\frac{\lambda_k}{\lambda_{k+1}})^b \widetilde{p}_{vk}(n+b, Z^*) + \widetilde{p}_{v,k+1}(n+b, Z^*)] \\ &\qquad + \widetilde{p}_{v,k+2}(n+b, Z^*) \\ &\succ_{coe} s_k(n+b)\widetilde{p}_{vk}(n+b, Z^*) + \widetilde{p}_{v,k+1}(n+b, Z^*) \\ &\qquad + \widetilde{p}_{v,k+2}(n+b, Z^*) \\ &\succ_{coe} 0, \end{aligned} \tag{32}$$

for any $b \geq B'_{vk}$. So, we have

$$\begin{aligned} &(\frac{\lambda_{k+1}}{\lambda_{k+2}})^b [s_k(n)(\frac{\lambda_k}{\lambda_{k+1}})^b \widetilde{p}_{vk}(n+b, Z^*) + \widetilde{p}_{v,k+1}(n+b, Z^*)] \\ &\qquad + \widetilde{p}_{v,k+2}(n+b, Z^*) \succ_{coe} 0. \end{aligned} \tag{33}$$

Multiplying both sides of the above formula by λ_{k+2}^b, we get

$$\begin{aligned} &s_k(n)\lambda_k^b \widetilde{p}_{vk}(n+b, Z^*) + \lambda_{k+1}^b \widetilde{p}_{v,k+1}(n+b, Z^*) \\ &\qquad + \lambda_{k+2}^b \widetilde{p}_{v,k+2}(n+b, Z^*) \succ_{coe} 0. \end{aligned} \tag{34}$$

Hence, by Equation (12), it follows that

$$\begin{aligned}\widetilde{H}_{v,k,k+2}(n, A^b Z^*) = s_k(n)\widetilde{p}_{vk}(n, A^b Z^*) + \widetilde{p}_{v,k+1}(n, A^b Z^*)\\ + \widetilde{p}_{v,k+2}(n, A^b Z^*) \succ_{coe} 0\end{aligned}$$

for any $b \geq B'_{vk}$. Thus, for any $b \geq B'_{vk}$, $A^b Z^* \in S_{\widetilde{H}_{v,k,k+2}}$. In the same way, we can get that for any $b \geq B'_{vk}$, $A^b Z^* \in S_{\widetilde{H}_{v,k,u}}$ for all $k+1 \leq u \leq t$. So, for any $b \geq B'_{vk}$, $A^b Z^* \in NT^o_{v,k,q_{vk}} \subseteq NT^o_v$.

□

Theorem 8. *With the above notion. If $NT^o_v \neq \emptyset$, then $NT_v \neq \emptyset$.*

Proof. To prove this, we will claim that $NT^o_v \subseteq NT_v$. Since $NT^o_v \neq \emptyset$, we take arbitrarily a point Z^* from NT^o_v,

$$Z^* \in \bigcup_{k=1}^{t} \bigcup_{q_{vk}=0}^{d_{vk}} (T_{v,k-1} \bigcap S_{\widetilde{p}_{vk,q_{vk}}} \bigcap (\bigcap_{u=k+1}^{t} S_{\widetilde{H}_{v,k,u}})).$$

Without loss of generality, assume that

$$Z^* \in NT^o_{v,k,q_{vk}} = T_{v,k-1} \bigcap S_{\widetilde{p}_{vk,q_{vk}}} \bigcap (\bigcap_{u=k+1}^{t} S_{\widetilde{H}_{v,k,u}}).$$

Firstly, since

$$Z^* \in S_{\widetilde{p}_{vk,q_{vk}}} \bigcap (\bigcap_{u=k+1}^{t} S_{\widetilde{H}_{v,k,u}}),$$

$\widetilde{p}_{v,k}(n, Z^*)$ is a polynomial of degree q_{vk} and each $\widetilde{H}_{v,k,u}(n, Z^*)$ is a polynomial of degree $\mu_k + q_{vk}$. And all the coefficients of $\widetilde{p}_{v,k}(n, Z^*)$, $\widetilde{H}_{v,k,u}(n, Z^*)$ are positive. That is, $\widetilde{p}_{v,k}(n, Z^*) \succ_{coe} 0$, and for each $u \in \overline{[k+1, t]}$, $\widetilde{H}_{v,k,u}(n, Z^*) \succ_{coe} 0$. By the definition of $s_k(n)$, we know

$$(\frac{\lambda_k}{\lambda_{k+1}})^n \succeq_{\mathbb{Z}_{\geq 0}} s_k(n) \succ_{\mathbb{Z}_{\geq 0}} 0.$$

So, since $\widetilde{p}_{v,k}(n, Z^*) \succ_{coe} 0$ implies that $\widetilde{p}_{v,k}(n, Z^*) \succ_{\mathbb{Z}_{\geq 0}} 0$, we have

$$(\frac{\lambda_k}{\lambda_{k+1}})^n \widetilde{p}_{v,k}(n, Z^*) \succeq_{\mathbb{Z}_{\geq 0}} s_k(n)\widetilde{p}_{v,k}(n, Z^*) \succ_{\mathbb{Z}_{\geq 0}} 0. \quad (35)$$

Since $Z^* \in S_{\widetilde{H}_{v,k,u}}$, we also have

$$\widetilde{H}_{v,k,u}(n, Z^*) = \\ s_k(n)\widetilde{p}_{vk}(n, Z^*) + \widetilde{p}_{v,k+1}(n, Z^*) + \cdots + \widetilde{p}_{v,u}(n, Z^*) \succ_{coe} 0,$$

for all $k+1 \leq u \leq t$. Hence,

$$\widetilde{H}_{v,k,u}(n, Z^*) = \\ s_k(n)\widetilde{p}_{vk}(n, Z^*) + \widetilde{p}_{v,k+1}(n, Z^*) + \cdots + \widetilde{p}_{v,u}(n, Z^*) \succ_{\mathbb{Z}_{\geq 0}} 0,$$

for all $k+1 \leq u \leq t$. Therefore, we get

$$s_k(n)\widetilde{p}_{v,k}(n, Z^*) + \widetilde{p}_{v,k+1}(n, Z^*) \succ_{\mathbb{Z}_{\geq 0}} 0.$$

Thus, by Formula (35), we get

$$(\frac{\lambda_k}{\lambda_{k+1}})^n\widetilde{p}_{v,k}(n, Z^*) + \widetilde{p}_{v,k+1}(n, Z^*) \succeq_{\mathbb{Z}_{\geq 0}} \\ s_k(n)\widetilde{p}_{v,k}(n, Z^*) + \widetilde{p}_{v,k+1}(n, Z^*) \succ_{\mathbb{Z}_{\geq 0}} 0.$$

Clearly, since $(\frac{\lambda_{k+1}}{\lambda_{k+2}}) > 1$, it follows that

$$\begin{aligned}(\frac{\lambda_{k+1}}{\lambda_{k+2}})^n[(\frac{\lambda_k}{\lambda_{k+1}})^n\widetilde{p}_{v,k}(n, Z^*) + \widetilde{p}_{v,k+1}(n, Z^*)] \\ &\succ_{\mathbb{Z}_{\geq 0}} (\frac{\lambda_k}{\lambda_{k+1}})^n\widetilde{p}_{v,k}(n, Z^*) + \widetilde{p}_{v,k+1}(n, Z^*) \\ &\succeq_{\mathbb{Z}_{\geq 0}} s_k(n)\widetilde{p}_{v,k}(n, Z^*) + \widetilde{p}_{v,k+1}(n, Z^*) \\ &\succ_{\mathbb{Z}_{\geq 0}} 0.\end{aligned}$$

At the same time, because

$$\widetilde{H}_{v,k,k+2}(n, Z^*) = \\ s_k(n)\widetilde{p}_{vk}(n, Z^*) + \widetilde{p}_{v,k+1}(n, Z^*) + \widetilde{p}_{v,k+2}(n, Z^*) \succ_{\mathbb{Z}_{\geq 0}} 0,$$

we have

$$(\frac{\lambda_{k+1}}{\lambda_{k+2}})^n[(\frac{\lambda_k}{\lambda_{k+1}})^n\widetilde{p}_{v,k}(n, Z^*) + \widetilde{p}_{v,k+1}(n, Z^*)] \\ + \widetilde{p}_{v,k+2}(n, Z^*) \succ_{\mathbb{Z}_{\geq 0}} 0.$$

In the same way, we can get

$$
\begin{aligned}
(\frac{\lambda_{t-1}}{\lambda_t})^n(\cdots((\frac{\lambda_{k+1}}{\lambda_{k+2}})^n\cdot((\frac{\lambda_k}{\lambda_{k+1}})^n\widetilde{p}_{v,k}(n,Z^*)+\widetilde{p}_{v,k+1}(n,Z^*)) \\
+\widetilde{p}_{v,k+2}(n,Z^*)+\cdots+\widetilde{p}_{v,t-1}(n,Z^*))+\widetilde{p}_{v,t}(n,Z^*) \\
\succ_{\mathbb{Z}\geq 0} s_k(n)\widetilde{p}_{v,k}(n,Z^*)+\widetilde{p}_{v,k+1}(n,Z^*)+\cdots \\
+\widetilde{p}_{v,t}(n,Z^*) \\
\succ_{\mathbb{Z}\geq 0} 0.
\end{aligned} \tag{36}
$$

Multiplying both sides of (36) by λ_t^n, we get

$$
\begin{aligned}
\lambda_k^n\widetilde{p}_{v,k}(n,Z^*)+\lambda_{k+1}^n\widetilde{p}_{v,k+1}(n,Z^*) \\
+\cdots+\lambda_t^n\widetilde{p}_{v,t}(n,Z^*)\succ_{\mathbb{Z}\geq 0} 0.
\end{aligned}
$$

Moreover, since $Z^*\in T_{v,k-1}$, we have

$$\widetilde{p}_{v,1}(n,Z^*)=\widetilde{p}_{v,k-1}(n,Z^*)\equiv 0,$$

and we get

$$
\begin{aligned}
Cond_v(n,Z^*)=\lambda_1^n\cdot 0+\lambda_2^n\cdot 0+\cdots+\lambda_{k-1}^n\cdot 0 \\
+\lambda_k^n\widetilde{p}_{vk}(n,Z^*)+\cdots+\lambda_t^n\widetilde{p}_{vt}(n,Z^*)\succ_{\mathbb{Z}_{\geq 0}} 0.
\end{aligned} \tag{37}
$$

This implies that $Z^*\in NT_v$. Hence, since Z^* is taken arbitrarily, it immediately follows that $NT_v^o\subseteq NT_v$. This completes this proof of the theorem. □

The result below immediately follows from Theorem 7 and Theorem 8.

Theorem 9. *With the above notion. We have $NT_v\neq\emptyset$ if and only if $NT_v^o\neq\emptyset$.*

By the proof of Theorem 7, we know that for any $Z^*\in NT_v$, there must exist $B'_{vk}=max(B^o_{vk},B^*_{vk})\in\mathbb{Z}_{\geq 0}$ such that

$$A^bZ^*\in NT^o_{v,k,q_{vk}}\subseteq NT^o_v.$$

for any $b\geq B'_{vk}$. Note that in general, the value of B'_{vk} depends on the choice of Z^*, since the value of B^o_{vk} depends on the choice of Z^*. However, it is easy to show that if Z^* is taken from $NT^o_{v,k,q_{vk}}$, then

$$B'_{vk}=max(B^o_{vk},B^*_{vk})=max(0,B^*_{vk})=B^*_{vk}.$$

And by Proposition 4, we know that the value of B^*_{vk} does not depend on the choice of Z^*, but only on $s_k(n)$ and $\frac{\lambda_k}{\lambda_{k+1}}$. Therefore, for any $Z^* \in NT^o_{v,k,q_{vk}}$, there must exist $B^*_{vk} \in \mathbb{Z}_{\geq 0}$, which is independent of the choice of Z^*, such that $A^b Z^* \in NT^o_{v,k,q_{vk}}$ for any $b \geq B^*_{vk}$. In other words, there exists $B^*_{vk} \in \mathbb{Z}_{\geq 0}$, such that for any point $Z^* \in NT^o_{v,k,q_{vk}}$, $A^b Z^* \in NT^o_{v,k,q_{vk}}$ for any $b \geq B^*_{vk}$. By the above arguments, we have the following results. And some notations in the proof of Theorem 7 will be continuously used.

Theorem 10. *With the above notion. There exists a positive integer B^*_{vk} such that for any $b \geq B^*_{vk}$, $NT^o_{v,k,q_{vk}}$ is A^b-invariant.*

Proof. To prove this, we will show that there must exist $B^*_{vk} \in \mathbb{Z}_{\geq 0}$ such that for any point $Z^* \in NT^o_{v,k,q_{vk}}$, $A^b Z^* \in NT^o_{v,k,q_{vk}}$ for any $b \geq B^*_{vk}$. Take any point Z^* from $NT^o_{v,k,q_{vk}}$. By the proof of Theorem 7, we know that there must exist $B'_{vk} = max(B^o_{vk}, B^*_{vk}) \in \mathbb{Z}_{\geq 0}$ such that for any $b \geq B'_{vk}$, $A^b Z^* \in NT^o_{v,k,q_{vk}}$. Next, we just need to show that $B'_{vk} = B^*_{vk}$. Where B^*_{vk} is defined as in the proof of Theorem 7. First of all, since $Z^* \in NT^o_{v,k,q_{vk}}$, all the coefficients of $\widetilde{p}_{vk}(n, Z^*)$ and $\widetilde{H}_{v,k,u}(n, Z^*)$ are positive, respectively. So, in the proof of Theorem 7, we can take $b_{q_{vk}} = 0$ and $b_{\widetilde{H}_{v,k,u}} = 0$, for all $k+1 \leq u \leq t$. And then,

$$\begin{aligned} B^o_{vk} &= \max(\max_{u=k+1}^{t}\{b_{\widetilde{H}_{vku}}\}, b_{q_{vk}}) \\ &= \max(\max_{u=k+1}^{t}\{0\}, 0) = 0. \end{aligned}$$

Thus, $B'_{vk} = \max(B^o_{vk}, B^*_{vk}) = \max(0, B^*_{vk}) = B^*_{vk}$. Note that the value of B^*_{vk} does not depend on the choice of Z^*, but only on $s_k(n)$ and $\frac{\lambda_k}{\lambda_{k+1}}$, according to Proposition 4. Therefore, there must exist $B^*_{vk} \in \mathbb{Z}_{\geq 0}$, such that for any $Z^* \in NT^o_{v,k,q_{vk}}$, we have $A^b Z^* \in NT^o_{v,k,q_{vk}}$ for any $b \geq B^*_{vk}$.

Finally, let us consider the special case when $k = t$, i.e.,

$$\begin{aligned} NT^o_{v,t,q_{vt}} &= T_{v,t-1} \bigcap S_{\widetilde{p}_{vt},q_{vt}} \bigcap (\bigcap_{u=t+1}^{t} S_{\widetilde{H}_{v,t,u}}) \\ &= T_{v,t-1} \bigcap S_{\widetilde{p}_{vt},q_{vt}} \bigcap \mathbb{R}^r \\ &= T_{v,t-1} \bigcap S_{\widetilde{p}_{vt},q_{vt}}. \end{aligned}$$

By the above arguments, it is easy to see that $NT^o_{v,t,q_{vt}}$ is A^b-invariant for any $b \geq 0$. So, $B^*_{vt} = 0$. We now complete the proof of the theorem. □

In terms of the proof of Theorem 8, we have $NT_v^o \subseteq NT_v$. This directly implies that

$$\bigcap_{v=1}^{s} NT_v^o \subseteq \bigcap_{v=1}^{s} NT_v.$$

Moreover, by the above proof of Theorem 7, we know that for any point $Z^* \in NT_v$, there exists $B'_{vk} \in \mathbb{Z}_{\geq 0}$ such that $A^b Z^* \in NT_v^o$ for any $b \geq B'_{vk}$. Therefore, This suggests that each point $Z^* \in \bigcap_{v=1}^{s} NT_v$ must fall into $\bigcap_{v=1}^{s} NT_v^o$ after $\max_{v=1}^{s} B'_{vk}$ iterations. And the above results established for the termination of Program $\mathfrak{U}_v$ can be naturally generalized to the termination of Program $\mathfrak{J}_1$, as follows.

Corollary 6. *With the above notion. For Program $\mathfrak{J}_1$,*

$$NT = \bigcap_{v=1}^{s} NT_v \neq \emptyset \textit{ if and only if } \bigcap_{v=1}^{s} NT_v^o \neq \emptyset.$$

Let

$$\widehat{B} = \max_{v=1}^{s} \max_{k=1}^{t-1} B^*_{vk}. \tag{38}$$

In fact, since B^*_{vk} is just related to $s_k(n)$ and $\frac{\lambda_k}{\lambda_{k+1}}$, we have

$$B^*_{1k} = B^*_{2k} = \cdots = B^*_{sk}.$$

Especially, set $B^*_{1t} = B^*_{2t} = \cdots = B^*_{st} = 0$.

According to Theorem 10, it is easy to see that $\bigcap_{v=1}^{s} NT^o_{v,k_v,q_{vk_v}}$ is $A^{\widehat{B}}$-invariant. And if $\bigcap_{v=1}^{s} NT^o_{v,k_v,q_{vk_v}} \neq \emptyset$, then there must exist a nonempty $A^{\widehat{B}}$-invariant subset in the region specified by $BZ > 0$. Therefore, the following result is established. For convenience, some notations are given firstly. Let $\widetilde{a}_{v,k,l}(Z) = \widetilde{\boldsymbol{a}}_{v,k,l} \cdot Z$ and $\widetilde{\beta}_{v,k,u,l}(Z) = \widetilde{\boldsymbol{\beta}}_{v,k,u,l} \cdot Z$, since $\widetilde{a}_{v,k,l}(Z)$'s and $\widetilde{\beta}_{v,k,u,l}(Z)$'s are all linear. Let $L_{q_{vk}} = \overline{[0, r_\perp - 1]}$ and $L_k = \overline{[1, t]}$. Let $L = L_k \times L_{q_{vk}}$. Where $r_\perp \geq r_0 = \max_{i=1}^{m} r_i$. Let

$$V_{vk} = (\widetilde{\boldsymbol{a}}_{v,k,0}, ..., \widetilde{\boldsymbol{a}}_{v,k,r_\perp - 1})^T$$

and

$$V_{v,k,u} = (\widetilde{\boldsymbol{\beta}}_{v,k,u,0}, ..., \widetilde{\boldsymbol{\beta}}_{v,k,u,\mu_k + q_{vk} - 1})^T.$$

By Formula (23), the following result can be established.

Theorem 11. *With the above notion. Program $\mathfrak{J}_1$ is non-terminating if and only if there exists an s-tuple $(u_{k_1,q_{1k_1}}, ..., u_{k_s,q_{sk_s}}) \in L^s$, such that the following sentence (39) is true in the theory of reals. Where $u_{k_j,q_{jk_j}} = (k_j, q_{jk_j})$.*

$$\begin{aligned}
&\exists V_{11} \cdots \exists V_{1k_1} \exists V_{1,k_1,k_1+1} \cdots \exists V_{1,k_1,t}, \cdots, \exists V_{s1} \cdots \exists V_{sk_s} \exists V_{s,k_s,k_s+1} \cdots \exists V_{s,k_s,t} \\
&[\exists Z.(\bigwedge_{v=1}^{s}(\phi_{T_{v,k_v-1}}(V_{v1}, ..., V_{v,k_v-1}, Z) \wedge \phi_{S_{\widetilde{p}_{vk_v},q_{vk_v}}}(V_{vk_v}, Z) \\
&\wedge \bigwedge_{u=k_v+1}^{t} \phi_{\widetilde{H}_{v,k_v,u}}(V_{v,k_v,u}, Z)) \wedge \bigwedge_{j=0}^{\widehat{B}-1} BA^j Z > 0) \wedge \forall Z.(\\
&\bigwedge_{v=1}^{s}(\phi_{T_{v,k_v-1}}(V_{v1}, ...V_{v,k_v-1}, Z) \wedge \phi_{S_{\widetilde{p}_{vk_v},q_{vk_v}}}(V_{vk_v}, Z) \\
&\wedge \bigwedge_{u=k_v+1}^{t} \phi_{\widetilde{H}_{v,k_v,u}}(V_{v,k_v,u}, Z)) \wedge \bigwedge_{j=0}^{\widehat{B}-1} BA^j Z > 0 \\
&\Rightarrow (\bigwedge_{v=1}^{s}(\phi_{T_{v,k_v-1}}(V_{v1}, ...V_{v,k_v-1}, A^{\widehat{B}} Z) \wedge \phi_{S_{\widetilde{p}_{vk_v},q_{vk_v}}}(V_{vk_v}, A^{\widehat{B}} Z) \\
&\wedge \bigwedge_{u=k_v+1}^{t} \phi_{\widetilde{H}_{v,k_v,u}}(V_{v,k_v,u}, A^{\widehat{B}} Z)) \wedge \bigwedge_{j=0}^{\widehat{B}-1} BA^{j+\widehat{B}} Z > 0))]
\end{aligned} \tag{39}$$

Especially, for the sake of completeness, when $k_v = t$, let

$$\bigwedge_{u=t+1}^{t} \phi_{\widetilde{H}_{v,k_v,u}}(V_{v,k_v,u}, Z) = \textbf{\textit{TRUE}},$$

and when $k_v = 1$, let

$$\phi_{T_{v,0}}(V_{v1}, ..., V_{v,k_v-1}, Z) = \textbf{\textit{TRUE}}.$$

Where

$$\phi_{T_{v,k_v-1}} \triangleq \bigwedge_{j=1}^{k_v-1} \bigwedge_{l=0}^{r_\perp-1} \widetilde{\boldsymbol{a}}_{vjl} \cdot \boldsymbol{Z} = 0,$$

$$\phi_{S_{\widetilde{p}_{vk_v},q_{vk_v}}} \triangleq \bigwedge_{j>q_{vk_v}} \widetilde{\boldsymbol{a}}_{v,k_v,j} \cdot \boldsymbol{Z} = 0 \wedge \bigwedge_{j\leq q_{vk_v}} \widetilde{\boldsymbol{a}}_{v,k_v,j} \cdot \boldsymbol{Z} > 0,$$

$$\phi_{\widetilde{H}_{v,k_v,u}} \triangleq \bigwedge_{l<\mu_k+q_{vk_v}} \widetilde{\boldsymbol{\beta}}_{v,k_v,u,l} \cdot \boldsymbol{Z} > 0,$$

$$0 \leq k_v \leq t,\, 0 \leq q_{v,k_v} \leq r_\perp - 1,\ k_v + 1 \leq u \leq t.$$

Proof. The proof will be similar to the argument given in Theorem 4. First of all, let $\Omega_0 = \Omega = \{Z \in \mathbb{R}^r : BZ > 0\}$ and $\Omega_{\widehat{B}-1} = \{\mathbb{R}^r : BZ > 0, ..., BA^{\widehat{B}-1}Z > 0\}$. It is not difficult to notice that

$$\bigcap_{v=1}^{s} NT_v^o = \bigcup_{(u_{k_1,q_{1k_1}},...,u_{k_s,q_{sk_s}})\in L^s} \bigcap_{v=1}^{s} NT_{v,k_v,q_{vk_v}}^o$$

and

$$\bigcap_{v=1}^{s} NT_v^o \subseteq NT \subseteq \Omega_{\widehat{B}-1} \subseteq \Omega.$$

Hence, $\bigcap_{v=1}^s NT_v^o \cap \Omega_{\widehat{B}-1} = \bigcap_{v=1}^s NT_v^o$, is $A^{\widehat{B}}$-invariant, since $\bigcap_{v=1}^s NT_{v,k_v,q_{vk_v}}^o$ is $A^{\widehat{B}}$-invariant. So, if Program $\mathfrak{J}_1$ is non-terminating, then

$$\bigcap_{v=1}^{s} NT_v^o = \bigcap_{v=1}^{s} NT_v^o \cap \Omega_{\widehat{B}-1} \neq \emptyset.$$

That is, there must exist an s-tuple

$$(u_{k_1,q_{1k_1}}, u_{k_2,q_{2k_2}}, ..., u_{k_s,q_{sk_s}}) \in L^s,$$

such that

$$\bigcap_{v=1}^{s} NT_{v,k_v,q_{vk_v}}^o \cap \Omega_{\widetilde{B}-1} \neq \emptyset.$$

It immediately follows that Formula (39) must be true.

Conversely, if there exists an s-tuple

$$e = (u_{k_1,q_{1k_1}}, u_{k_2,q_{2k_2}}, ..., u_{k_s,q_{sk_s}}) \in L^s,$$

such that Formula (39) is true, then we know that there must exist a nonempty $A^{\widehat{B}}$-invariant set $\hat{S}_e$ in Ω, which can be characterized by the following formula

$$\begin{aligned}\hat{S}_e \triangleq & \bigwedge_{v=1}^{s}(\phi_{T_{v,k_v-1}}(V_{v1},...,V_{v,k_v-1},Z) \\ & \wedge \phi_{\hat{S}_{\widetilde{p}_{vk_v},q_{vk_v}}}(V_{vk_v},Z) \\ & \wedge \bigwedge_{u=k_v+1}^{t} \phi_{\widetilde{H}_{v,k_v,u}}(V_{v,k_v,u},Z)) \wedge \bigwedge_{j=0}^{\widehat{B}-1} BA^j Z > 0.\end{aligned}$$

Clearly, for arbitrary $Z^* \in \hat{S}_e \subseteq \Omega$, we have

$$A^{\widehat{B}} Z^* \in \hat{S}_e \subseteq \Omega$$

and

$$A^j Z^* \in \Omega,$$

for all $j = 0, ..., \widehat{B} - 1$, since $Z^* \in \hat{S}_e \subseteq \Omega_{\widehat{B}-1}$. Therefore, for any $j \in \mathbb{Z}_{\geq 0}$, $A^j Z^* \in \Omega$. Thus, Z^* is exactly a non-terminating point of Program $\mathfrak{J}_1$. Obviously, this implies that $\mathfrak{J}_1$ is non-terminating. This completes the proof of the theorem. □

Remark 4. It is easy to see that Formula (39) can be constructed directly, if the values of t and $\widehat{B}$, the degree μ_k of $s_k(n)$ and the upper bound $r_\perp$ for the degrees d_{vk}'s of $\widetilde{p}_{v,k}(n, Z)$'s w.r.t n can be obtained in advance. Where t denotes the number of distinct positive eigenvalues of the assignment matrix A of $\mathfrak{J}_1$. Since the assignment matrices of $\mathfrak{J}_1$ and $\mathbf{P}_1$ have the same positive eigenvalues, the value of t can be computed by applying the method in [18, 19] to the assignment matrix $\widetilde{A}$ of $\mathbf{P}_1$. In Formula (14), we know that the degree d_{vk} of $\widetilde{p}_{vk}(n, Z)$ w.r.t n is less than the maximal dimension of Jordan blocks having the same eigenvalue $\lambda_k(\lambda_k > 0)$. Denote by $mul(\lambda_k)$ the algebraic multiplicity of λ_k. Let

$$r_\perp = \max_{j=1}^{t}\{mul(\lambda_j)\}, (\lambda_j > 0). \tag{40}$$

Clearly, $r_\perp$ is exactly the desired common upper bound for d_{vk}'s, since the dimensions of Jordan blocks having the eigenvalues λ_k must be less than or equal to the algebraic multiplicity of λ_k. The algebraic multiplicity of each positive eigenvalue λ_k can be computed by applying the method given in [19, 18] to $\widetilde{A}$. Therefore, Theorem 11 suggests another method to check the termination of Program $\mathbf{P}_1$.

Corollary 6 and Theorem 11 suggest two different methods for determining the termination of Program $\mathfrak{J}_1$ with general structure. The method due to Corollary 6 depends on the computation of $s_k(n)$ and $\widetilde{p}_{vk}(n, Z)$'s. But the method due to Theorem 11 depends only on the estimate of the common upper bound $r_\perp$ of the degrees d_{vk}'s of $\widetilde{p}_{vk}(n, Z)$'s, and the construction of $s_k(n)$, the number t of distinct positive eigenvalues and $\widehat{B}$. By the definition of $\widehat{B}$, we know that the computation of $\widehat{B}$ can be reduced to the computation of B^*_{vk}. Next, we will present the methods for computing $s_k(n)$ and B^*_{vk}.

The construction of $s_k(n)$. To construct $NT_{v,k,q_{vk}}$ as above, the key step is to construct $s_k(n)$, which is the lower bound of $(\frac{\lambda_k}{\lambda_{k+1}})^n$ and satisfies the two conditions below:

- all its coefficients s_{kj}'s are positive;
- $\mu_k \geq \max_{i=1}^m r_i$;

Proposition 5 tells us that such $s_k(n)$ can be directly constructed in the following form:

$$s_k(n) = \frac{1}{\mathcal{M}_k(b^*)}\mathcal{M}_k(n).$$

Where $\mathcal{M}_k(n)$ is defined as in Proposition 5. Thus, to compute $s_k(n)$, the key step is to find the value of b^*. A method for computing b^* is given in the proof of Proposition 5. Next, we will illustrate this method in detail and some necessary notations in Proposition 5 will be continuously used. Recall that $\epsilon_k = \frac{\lambda_k}{\lambda_{k+1}} - 1$ and $\mathcal{H}_k(n) = \sum_{j=0}^{\mu_k+1} C_n^j \epsilon_k^j = \sum_{j=0}^{\mu_k+1} h_j(\epsilon_k) n^j$. Because $\lambda_1 > \cdots > \lambda_t > 0$, we have $\epsilon_k > 0$. Since μ_k is required to be a positive integer greater than or equal to $\max_{i=1}^m r_i$ and $\max_{i=1}^m r_i \leq \max_{j=1}^t \lambda_j$, without loss of generality, we can take

$$\mu_k = r_\perp = \max_{j=1}^{t}\{mul(\lambda_j)\}, (\lambda_j > 0). \tag{41}$$

Let $\widetilde{\mathcal{H}}_k(n) = \mathcal{H}_k(n) - \mathcal{M}_k(n) = \sum_{j=0}^{\mu_k+1} \widetilde{h}_j(\epsilon_k)n^j$. Clearly, $\widetilde{H}_k(n)$ is a polynomial of degree $\mu_k + 1$ and its the leading coefficient is positive. Let

$$\hat{\mathcal{H}}_k(n) = \sum_{j=0}^{\mu_k+1} \beta_j(\epsilon_k, b)n^j = \widetilde{\mathcal{H}}_k(n+b).$$

Therefore, in terms of Proposition 1, construct the following semi-algebraic system:

$$\mathfrak{S}_+ = \{\beta_{\mu_k+1}(\epsilon_k, b) > 0, \cdots, \beta_0(\epsilon_k, b) > 0, b \geq 0\}.$$

Solving the above semi-algebraic system, we can get the desired b^*.

In general, we can not compute the accurate eigenvalue λ_k of the assignment matrix A. This indicates that in general case, ϵ_k can not be computed exactly too. But, ϵ_k can be represented by its minimal polynomial $Q_{\epsilon_k}(\alpha_k)$ over the rationals and an isolating interval I_{ϵ_k}, since ϵ_k is an algebraic number. Thus, to compute b^* is equivalent to solve the semi-algebraic system

$$Q_{\epsilon_k}(\alpha_k) = 0 \wedge \bigwedge_{j=0}^{\mu_k+1} \beta_j(\alpha_k, b) > 0 \wedge b \geq 0 \wedge I_{\alpha_k}. \quad (42)$$

By the arguments mentioned above, given the minimal polynomial and an isolating interval for ϵ_k, we can compute the desired b^*. This enables us to construct $s_k(n)$.

The computation of B^*_{vk}.

In the proof of Theorem 7, B^*_{vk} is introduced to guarantee that Formula (26) holds. According to the proof of Proposition 4, we next present a method for computing B^*_{vk}. For convenience, some notations in Proposition 4 are continuously used below. Let $\delta_k = \frac{\lambda_k}{\lambda_{k+1}} = (1+\epsilon_k)(\epsilon_k > 0)$ and $s_k(n) = \sum_{j=0}^{\mu_k} s_{kj}n^j$. Let $s_k(n+b) = \sum_{j=0}^{\mu_k} \beta_{kj}(b)n^j$ and $s_k(n)\delta_k^b = \sum_{j=0}^{\mu_k}(s_{kj}\delta_k^b)n^j$. Clearly, because $s_{kj} > 0$, $s_{kj}\delta_k^b \gg \beta_{kj}(b)$ as $b \to +\infty$, for all $0 \leq j \leq \mu_k$. Where $\beta_{kj}(b)$ is a polynomial in b of degree $\mu_k - j$. Thus, there must exist $B^*_{kj1} \geq 0$, such that $s_{kj}\delta_k^b \gg \beta_{kj}(b)$ for any $b \geq B^*_{kj1}$. Such B^*_{kj1} can be computed by Proposition 1 as follows. First, let

$$w_{kj}(b) = s_{kj} \sum_{i=0}^{\mu_k-j+1} \binom{b}{i} \epsilon_k^i = \sum_{i=0}^{\mu_k-j+1} w_{kj,i}(\epsilon_k)b^i.$$

be a polynomial in b of degree $\mu_k - j + 1$, whose leading coefficient is positive. It is not difficult to see that $w_{kj}(b)$ is the lower bound of $s_{kj}(\frac{\lambda_k}{\lambda_{k+1}})^b$, i.e.,

$$s_{kj}\delta_k^b = s_{kj}(\frac{\lambda_k}{\lambda_{k+1}})^b = s_{kj}(1+\epsilon_k)^b \geq w_{kj}(b)$$

for all $b \in \mathbb{Z}_{\geq 0}$. Let

$$u_{kj}(b) = w_{kj}(b) - \beta_{kj}(b) = \sum_{i=0}^{\mu_k - j+1} u_{kj,i}(\epsilon_k, b) b^i$$

and let

$$\hat{u}_{kj}(b) = u_{kj}(b + b') = \sum_{i=0}^{\mu_k - j+1} \beta_{kji}(\epsilon_k, b') b^i,$$

for $0 \leq j \leq \mu_k$. Obviously, $u_{kj}(b)$ has the same degree and the same leading coefficient as $w_{kj}(b)$, since the degree of $w_{kj}(b)$ is greater than that of $\beta_{kj}(b)$. Thus, since the leading coefficient of $u_{kj}(b)$ is positive, by Proposition 1, there must exist a positive number B_{kj1}^* such that for any $b \geq B_{kj1}^*$, all the coefficients of $u_{kj}(b)$ are positive. This implies that $u_{kj}(b) > 0$ for any $b \geq B_{kj1}^*$. Hence, we have

$$s_{kj}\delta_k^b > \beta_{kj}(b)$$

for any $b \geq B_{kj1}^*$. It immediately follows that

$$s_k(n)\delta_k^b \succ_{coe} s_k(n+b)$$

for any $b \geq B_{k1}^* = \max_{j=0}^{\mu_k} B_{kj1}^*$. Furthermore, the value of B_{kj1}^* can be computed by solving the following semi-algebraic system

$$\bigwedge_{i=0}^{\mu_k - j+1} u_{kj,i}(\alpha_k, b') > 0 \wedge b' \geq 0 \wedge Q_{\epsilon_k}(\alpha_k) = 0 \wedge I_{\epsilon_k}$$

As defined in Formula (42), $Q_{\epsilon_k}(\alpha_k)$ and I_{ϵ_k} are used to characterize ϵ_k. In addition, by the definition of $s_k(n)$, we know that all the coefficients of $s_k(n)$ are positive. Thus, $s_k(n+b) \succ_{coe} 0$ for any $b \in \mathbb{Z}_{\geq 0}$. So, take $B_{k2}^* = 0$. Let $B_k^* = \max(B_{k1}^*, B_{k2}^*) = B_{k1}^*$. Hence, take $B_{vk}^* = B_k^*$.

To sum up, when all the eigenvalues of the assignment matrix A of Program $\mathfrak{J}_1$ are the same, i.e., the assignment matrix $\widetilde{A}$ of Program $\mathbf{P}_1$ has only one positive eigenvalue, then Theorem 3-5 and Corollary 3-4 are available directly. Otherwise, Corollary 6 and Theorem 11 are utilized to determine the termination of Program $\mathfrak{J}_1$ with general structure.

4. Examples

In the section, we will take the following example to illustrate our methods established as above.

Example 3. Consider the below linear program

$$Q_4: \quad \textit{while} \;\; (\widetilde{B}\widetilde{X} > 0) \;\; \{\widetilde{X} := \widetilde{A}\widetilde{X}\}.$$

Where

$$\widetilde{B} = \begin{pmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \end{pmatrix}, \widetilde{A} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & 3 \end{pmatrix}, \widetilde{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

First, by computation, $\widetilde{A}$ has three simple real eigenvalues: $\lambda_1 = 3$, $\lambda_2 = 1$, $\lambda_3 = -2$. Thus, let $t = 2$ be the number of the distinct positive eigenvalues. Thus, computing the Jordan Canonical Form of $\widetilde{A}$, we get

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, P = \begin{pmatrix} -\frac{1}{2} & -1 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

Where P is an invertible matrix satisfying $J = P^{-1}\widetilde{A}P$. By Theorem 1, the termination of Q_4 can be reduced equivalently to that of the following program:

$$Q_5: \quad \textit{while} \;\; (\widetilde{B}PY > 0) \;\; \{Y := JY\}.$$

Where

$$\widetilde{B}P = \begin{pmatrix} 0 & 2 & -4 \\ -1 & -2 & 2 \end{pmatrix}, \; Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

By Theorem 2, since the termination of Q_5 is just related to the eigenspaces corresponding to the positive eigenvalues, the termination of Q_5 can be further reduced to that of Q_6 as follows:

$$Q_6: \quad \textit{while} \;\; (BZ > 0) \;\; \{Z := AZ\}$$

where

$$B = \begin{pmatrix} 2 & -4 \\ -2 & 2 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \triangleq \begin{pmatrix} y_2 \\ y_3 \end{pmatrix}.$$

Clearly, Q_6 is such a program as Program $\mathfrak{J}_1$.

By computation, we obtain that

$$\begin{aligned} Cond(n,Z) = BA^nZ &= \begin{pmatrix} \lambda_1^n\widetilde{p}_{11}(n,Z) + \lambda_2^n\widetilde{p}_{12}(n,Z) \\ \lambda_1^n\widetilde{p}_{21}(n,Z) + \lambda_2^n\widetilde{p}_{22}(n,Z) \end{pmatrix} \\ &= \begin{pmatrix} \lambda_2^n((\frac{\lambda_1}{\lambda_2})^n\widetilde{p}_{11}(n,Z) + \widetilde{p}_{12}(n,Z)) \\ \lambda_2^n((\frac{\lambda_1}{\lambda_2})^n\widetilde{p}_{21}(n,Z) + \widetilde{p}_{22}(n,Z)) \end{pmatrix} \end{aligned}$$

where, $\widetilde{p}_{11}(n,Z) = -4z_2$, $\widetilde{p}_{12}(n,Z) = 2z_1$, $\widetilde{p}_{21}(n,Z) = 2z_2$, and $\widetilde{p}_{22}(n,Z) = -2z_1$. Clearly, $d_{11} = d_{12} = d_{21} = d_{22} = 0$. Since $\lambda_1 = 3$ and $\lambda_2 = 1$, we have $\frac{\lambda_1}{\lambda_2} = 1 + \epsilon_1 = 3$. And because $mul(\lambda_1) = mul(\lambda_2) = 1$, by Formula (41), we get

$$\mu_1 = \max(mul(\lambda_1), mul(\lambda_2)) = 1.$$

By Proposition 5, we can get $s_1(n) = n + 1$. Thus, we have

$$\begin{aligned} &\widetilde{H}_{1,1,2} = s_1(n)\widetilde{p}_{11}(n,Z) + \widetilde{p}_{12}(n,Z) = -4\,z_2\,n - 4\,z_2 + 2\,z_1, \\ &\widetilde{H}_{2,1,2} = s_1(n)\widetilde{p}_{21}(n,Z) + \widetilde{p}_{22}(n,Z) = 2\,z_2\,n + 2\,z_2 - 2\,z_1, \\ &T_{10} = \mathbb{R}^2, T_{20} = \mathbb{R}^2, T_{11} = \{Z \in \mathbb{R}^2 :\ -4z_2 = 0\}, \\ &T_{21} = \{Z \in \mathbb{R}^2 :\ 2z_2 = 0\}, S_{\widetilde{p}_{11,0}} = \{Z \in \mathbb{R}^2 :\ -4z_2 > 0\}, \\ &S_{\widetilde{p}_{12,0}} = \{Z \in \mathbb{R}^2 :\ 2z_1 > 0\}, S_{\widetilde{p}_{21,0}} = \{Z \in \mathbb{R}^2 :\ 2z_2 > 0\}, \\ &S_{\widetilde{p}_{22,0}} = \{Z \in \mathbb{R}^2 :\ -2z_1 > 0\}, \\ &S_{\widetilde{H}_{1,1,2}} = \{Z \in \mathbb{R}^2 :\ -4z_2 + 2z_1 > 0\} \\ &S_{\widetilde{H}_{2,1,2}} = \{Z \in \mathbb{R}^2 :\ 2z_2 - 2z_1 > 0\}. \end{aligned}$$

Therefore, by Formula (22),

$$\begin{aligned} NT_v^o &= \bigcup_{k=1}^{t} \bigcup_{q_{vk}=0}^{d_{vk}} (T_{v,k-1} \bigcap S_{\widetilde{p}_{vk,q_{vk}}} \bigcap (\bigcap_{u=k+1}^{t} S_{\widetilde{H}_{v,k,u}})) \\ &= (T_{v0} \cap S_{\widetilde{p}_{v1,0}} \cap S_{\widetilde{H}_{v,1,2}}) \cup (T_{v1} \cap S_{\widetilde{p}_{v2,0}} \cap \mathbb{R}^2) \end{aligned} \tag{43}$$

By verifying, we find that

$$\begin{aligned} \bigcap_{v=1}^{2} NT_v^o &= \bigcap_{v=1}^{2} (T_{v0} \cap S_{\widetilde{p}_{v1,0}} \cap S_{\widetilde{H}_{v,1,2}}) \cup (T_{v1} \cap S_{\widetilde{p}_{v2,0}} \cap \mathbb{R}^2) \\ &= ((z_2 < 0 \wedge z_1 - 2z_2 > 0) \vee (z_2 = 0 \wedge z_1 > 0)) \\ &\quad \wedge ((z_2 > 0 \wedge z_2 - z_1 > 0) \vee (z_2 = 0 \wedge z_1 < 0)) \\ &= \emptyset. \end{aligned}$$

Hence, by Corollary 6, Program Q_6 is terminating. This implies that Program Q_4 is terminating too.

Next, we will apply Theorem 11 to determine the termination of Program Q_4. First, by Formula (41), we get

$$\begin{aligned}\mu_k = r_\perp &= \max_{j=1}^{t}\{mul(\lambda_j)\}\\ &= \max(mul(\lambda_1), mul(\lambda_2)) = 1\end{aligned}$$

In the example,

$$L_k = \overline{[1,t]} = \overline{[1,2]}, L_{q_{vk}} = \overline{[0, r_\perp - 1]} = \overline{[0,0]} = \{0\},$$

and $L = L_k \times L_{q_{vk}} = \{(1,0),(2,0)\}$. Thus,

$$\begin{aligned}L^s = L^2 = \{&((1,0),(1,0)),((1,0),(2,0)),\\ &((2,0),(1,0)),((2,0),(2,0))\}.\end{aligned}$$

Next, to find the value of $\widehat{B}$, we just need to compute B^*_{vk}. Because $s_1(n) = n+1$, by the arguments presented in the last part of Section 3, we know that $B^*_{v1} = 1$ such that

$$s_1(n)\delta_1^b = s_1(n)(\frac{\lambda_1}{\lambda_2})^b \succ_{coe} s_1(n+b) \succ_{coe} 0$$

for any $b \geq B^*_{vk}$. Thus, $\widehat{B} = \max(B^*_{11}, B^*_{21}) = 1$. Thus, by Theorem 11, we can construct the following four quantified formulas:

$$\begin{aligned}&\exists\widetilde{\boldsymbol{a}}_{110}\exists\widetilde{\boldsymbol{\beta}}_{1120}\exists\widetilde{\boldsymbol{a}}_{210}\exists\widetilde{\boldsymbol{\beta}}_{2120}\\ &[\exists Z.(\widetilde{\boldsymbol{a}}_{110}Z > 0 \wedge \widetilde{\boldsymbol{\beta}}_{1120}Z > 0 \wedge \widetilde{\boldsymbol{a}}_{210}Z > 0 \wedge \widetilde{\boldsymbol{\beta}}_{2120}Z > 0\\ &\wedge BZ > 0) \wedge \forall Z.((\widetilde{\boldsymbol{a}}_{110}Z > 0 \wedge \widetilde{\boldsymbol{\beta}}_{1120}Z > 0\\ &\wedge \widetilde{\boldsymbol{a}}_{210}Z > 0 \wedge \widetilde{\boldsymbol{\beta}}_{2120}Z > 0) \wedge BZ > 0\\ &\Rightarrow (\widetilde{\boldsymbol{a}}_{110}AZ > 0 \wedge \widetilde{\boldsymbol{\beta}}_{1120}AZ > 0 \wedge \widetilde{\boldsymbol{a}}_{210}AZ > 0 \wedge \widetilde{\boldsymbol{\beta}}_{2120}AZ > 0)\\ &\wedge BAZ > 0))]\end{aligned} \quad (44)$$

$$
\begin{aligned}
&\exists\widetilde{\boldsymbol{a}}_{110}\exists\widetilde{\boldsymbol{\beta}}_{1120}\exists\widetilde{\boldsymbol{a}}_{220}\exists\widetilde{\boldsymbol{a}}_{210}\\
&[\exists Z.(\widetilde{\boldsymbol{a}}_{110}Z > 0 \wedge \widetilde{\boldsymbol{\beta}}_{1120}Z > 0 \wedge \widetilde{\boldsymbol{a}}_{220}Z > 0 \wedge \widetilde{\boldsymbol{a}}_{210}Z = 0\\
&\wedge BZ > 0) \wedge \forall Z.((\widetilde{\boldsymbol{a}}_{110}Z > 0 \wedge \widetilde{\boldsymbol{\beta}}_{1120}Z > 0\\
&\wedge \widetilde{\boldsymbol{a}}_{220}Z > 0 \wedge \widetilde{\boldsymbol{a}}_{210}Z = 0) \wedge BZ > 0\\
&\Rightarrow (\widetilde{\boldsymbol{a}}_{110}AZ > 0 \wedge \widetilde{\boldsymbol{\beta}}_{1120}AZ > 0 \wedge \widetilde{\boldsymbol{a}}_{220}AZ > 0 \wedge \widetilde{\boldsymbol{a}}_{210}AZ = 0)\\
&\wedge BAZ > 0))]
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&\exists\widetilde{\boldsymbol{a}}_{110}\exists\widetilde{\boldsymbol{a}}_{120}\exists\widetilde{\boldsymbol{a}}_{210}\exists\widetilde{\boldsymbol{\beta}}_{2120}\\
&[\exists Z.(\widetilde{\boldsymbol{a}}_{110}Z = 0 \wedge \widetilde{\boldsymbol{a}}_{120}Z > 0 \wedge \widetilde{\boldsymbol{a}}_{210}Z > 0 \wedge \widetilde{\boldsymbol{\beta}}_{2120}Z > 0\\
&\wedge BZ > 0) \wedge \forall Z.((\widetilde{\boldsymbol{a}}_{110}Z = 0 \wedge \widetilde{\boldsymbol{a}}_{120}Z > 0\\
&\wedge \widetilde{\boldsymbol{a}}_{210}Z > 0 \wedge \widetilde{\boldsymbol{\beta}}_{2120}Z = 0) \wedge BZ > 0\\
&\Rightarrow (\widetilde{\boldsymbol{a}}_{110}AZ = 0 \wedge \widetilde{\boldsymbol{a}}_{120}AZ > 0 \wedge \widetilde{\boldsymbol{a}}_{210}AZ > 0 \wedge \widetilde{\boldsymbol{\beta}}_{2120}AZ = 0)\\
&\wedge BAZ > 0))]
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
&\exists\widetilde{\boldsymbol{a}}_{110}\exists\widetilde{\boldsymbol{a}}_{120}\exists\widetilde{\boldsymbol{a}}_{220}\exists\widetilde{\boldsymbol{a}}_{210}\\
&[\exists Z.(\widetilde{\boldsymbol{a}}_{110}Z = 0 \wedge \widetilde{\boldsymbol{a}}_{120}Z > 0 \wedge \widetilde{\boldsymbol{a}}_{220}Z > 0 \wedge \widetilde{\boldsymbol{\beta}}_{210}Z = 0\\
&\wedge BZ > 0) \wedge \forall Z.((\widetilde{\boldsymbol{a}}_{110}Z = 0 \wedge \widetilde{\boldsymbol{a}}_{120}Z > 0\\
&\wedge \widetilde{\boldsymbol{a}}_{220}Z > 0 \wedge \widetilde{\boldsymbol{\beta}}_{210}Z = 0) \wedge BZ > 0\\
&\Rightarrow (\widetilde{\boldsymbol{a}}_{110}AZ = 0 \wedge \widetilde{\boldsymbol{a}}_{120}AZ > 0 \wedge \widetilde{\boldsymbol{a}}_{220}AZ > 0 \wedge \widetilde{\boldsymbol{\beta}}_{210}AZ = 0)\\
&\wedge BAZ > 0))]
\end{aligned}
\tag{47}
$$

The satisfiability of the above four quantified formulae usually can be checked by Quantifier Elimination technique. By Farka's lemma, the four quantified formulae are verified to be false. Therefore, by Theorem 11, Program Q_4 is terminating.

5. Conclusion

In this paper, we reconsider the termination of Program $\mathbf{P}_1$. By Jordan Canonical Form technique, the termination of Program $\mathbf{P}_1$ can be reduced equivalently to that of Program $\mathfrak{J}_1$. For Program $\mathfrak{J}_1$, we construct a subset of the set NT of nonterminating points of Program $\mathfrak{J}_1$. This enables us to determine the termination of $\mathfrak{J}_1$ by checking if such a subset is empty. Thus, we get the first method for checking the termination of Program $\mathfrak{J}_1$. Furthermore, according to Theorem 10, such a subset is proven to be $A^{\widetilde{B}}$-invariant for a certain positive integer $\widetilde{B}$. This suggests the second method for checking the termination of $\mathfrak{J}_1$. Different from the first method, the second method is only involved with the information on positive eigenvalues of $\widetilde{A}$, such as the algebraic multiplicity of each positive eigenvalues.

The first method mentioned as above is also different from the existing methods for checking the termination of Program $\mathfrak{J}_1$. For example, in most cases, one point obtained by solving the semi-algebraic systems given by Tiwari, or Braverman, or Rebiha et al. is an N-nonterminating point, but not a non-terminating point. But, in contrast, any point in the semi-algebraic set given by our first method must be a non-terminating point, since such the semi-algebraic set is proven to be a subset of the set NT of non-terminating points of Program $\mathfrak{J}_1$. In addition, the first method is different from the method presented in our SNC paper, because the latter needs to reduce the termination of $\mathfrak{J}_1$ to the termination of two special class of linear programs $\mathfrak{J}_0$ and $\mathfrak{J}'_1$ in a recursive procedure.

Since our first method and the existing methods can be used to check the termination of Program $\mathfrak{J}_1$, they certainly can be applied to check the termination of Program $\mathbf{P}_1$ by Theorem 1 and Theorem 2. Clearly, for given a program such as Program $\mathbf{P}_1$, they all need to reduce the given program to a program such as $\mathfrak{J}_1$ by computing the Jordan canonical form of the assignment matrix of the given program. In other words, they all need to compute the Jordan canonical form of the assignment matrix of a program such as Program $\mathbf{P}_1$ firstly. However, in contrast, our second method due to Theorem 11 does not need to compute the Jordan canonical form of such a program as Program $\mathbf{P}_1$, and just needs to get the information on positive eigenvalues, such as the algebraic multiplicity of each positive eigenvalue, the number of all the distinct positive eigenvalues and so on. The reason is that the information on positive eigenvalues is enough to build Formula (39) in Theorem 11. It should be pointed out that the complexity of the second method may be very high, because too many parameters have to be introduced to build Formula (39).

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[1] K. Liu, Z. Shan, J. Wang, J. He, Z. Zhang, Y. Qin, Overview on major research plan of trustworthy software, Vol. 22, 2008, pp. 145–151(in Chinese with English abstract).

[2] L. Yang, N. Zhan, B. Xia, C. Zhou, Program verification by using discoverer, in: Verified Software: Theories, Tools, Experiments, Springer, 2008, pp. 528–538.

[3] L. Yang, C. Zhou, N. Zhan, B. Xia, Recent advances in program verification through computer algebra, Frontiers of Computer Science in China 4 (1) (2010) 1–16.

[4] A. R. Bradley, Z. Manna, H. B. Sipma, Linear ranking with reachability, in: Computer Aided Verification, Springer, 2005, pp. 491–504.

[5] Y. Chen, B. Xia, L. Yang, N. Zhan, C. Zhou, Discovering non-linear ranking functions by solving semi-algebraic systems, in: Theoretical Aspects of Computing–ICTAC 2007, Springer, 2007, pp. 34–49.

[6] M. A. Colón, H. B. Sipma, Synthesis of linear ranking functions, in: Tools and Algorithms for the Construction and Analysis of Systems, Springer, 2001, pp. 67–81.

[7] M. A. Colón, H. B. Sipma, Practical methods for proving program termination, in: Computer Aided Verification, Springer, 2002, pp. 442–454.

[8] P. Cousot, Proving program invariance and termination by parametric abstraction, lagrangian relaxation and semidefinite programming, in: Verification, Model Checking, and Abstract Interpretation, Springer, 2005, pp. 1–24.

[9] A. Podelski, A. Rybalchenko, A complete method for the synthesis of linear ranking functions, in: Verification, Model Checking, and Abstract Interpretation, Springer, 2004, pp. 239–251.

[10] A. Tiwari, Termination of linear programs, in: Computer Aided Verification, Springer, 2004, pp. 70–82.

[11] M. Braverman, Termination of integer linear programs, in: Computer aided verification, Springer, 2006, pp. 372–385.

[12] B. Xia, Z. Zhang, Termination of linear programs with nonlinear constraints, Journal of Symbolic Computation 45 (11) (2010) 1234–1249.

[13] B. Xia, L. Yang, N. Zhan, Z. Zhang, Symbolic decision procedure for termination of linear programs, Formal Aspects of Computing 23 (2) (2011) 171–190.

[14] R. Rebiha, N. Matrnge, A. Moura, Generating asymptotically non-terminant initial variable vallues for linear diagonalizable programs, in: Symbolic Computation in Software Science (SCSS'13), 2013, pp. 81–92.

[15] R. Rebiha, N. Matrnge, A. Moura, Generating asymptotically non-terminant initial variable vallues for linear programs, in: CoRR, 2014, p. abs/1407.4556.

[16] J. Ouaknine, J. Sousa Ponto, J. Worrell, On termination of integer linear loops, in: Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'15), SIAM, 2015, pp. 957–969.

[17] Y. Li, A recursive decision method for termination of linear programs, in: Proceedings of the 2014 Symposium on Symbolic-Numeric Computation, ACM, 2014, pp. 97–106.

[18] B. Xia, L. Yang, Some properties of the descrimination matrix of polynomials with applications, Acta Mathematicae Applicatae Sinica 26 (4) (2003) 652–663.

[19] L. Yang, Recent advances on determining the number of real roots of parametric polynomials, Journal of Symbolic Computation 28 (1) (1999) 225–242.