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Numerical Method for Computing the Stable Equilibrium of High Dimensional Predator Prey System

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Based on the numerical method for isolating the real root of semi-algebraic system and the Lyapunov's first method with the technique of linearization, an algorithm is presented to determine the stable positive equilibrium of high dimensional predator prey system.

Keywords: Homotopy Method, Real Root Isolation, Stability Analysis, Lyapunov's First Method.

1. INTRODUCTION

Solving polynomial system has been one of the central topics in computer algebra. It is required and used in many scientific and engineering applications. There are two basic skills for solving such polynomial system: symbolically and numerically. For symbolic method, there exits several methods, such as resultants,[1] Groebner bases[2] method introduced by Buchberger, and triangular sets.[3] Numerical calculation methods include Homotopy continuation Method,[4,5] Newton's iteration algorithm and interval bisection algorithm.

For many practical problems, in deed, we mainly care about the real roots of a polynomial system. For example, in this paper, we only consider the positive real equilibrium of the differential system. Based on the triangular set technique, Xia et al.[6] proposed an algorithm use Wu's method for isolating the real roots of a semi-algebraic system with integer coefficients, and made it more practical with interval algorithm in their later work.[7] Actually, these methods compute the exact results because they depend on symbolic computations, but they are restricted to small size systems because of the high complexity of the symbolic computation. In order to avoid this problem, Shen et al.[8] presented a numerical algorithm improving the efficiency based on homotopy continuation method combined with interval Newton iteration technique. Based on the algorithm, we proposed a numerical algorithm to computing the real roots of a semi-algebraic system in Ref. [9].

In many engineering applications, we need to create the corresponding mathematical model, and analysis its stability[10–12] using numerical method, or computing the value of parameter in the mathematical model.[13]

In most researches on biological system, analysis stability of equilibrium of a differential system is one of the central topic. Recently, Wang and Xia[14] propose a new and general approach for analyzing the stability of a large class of biological networks using real solving and solution classification. Luo and Lu[15] study the stability analysis for Lotka–Volterra systems. However, these algorithms only handle the system with low dimension because of the high complexity of the symbolic computation

For higher dimensional systems (dimension is not less than 5), we always qualitatively study the existence, uniqueness and stability of the positive equilibrium or periodic solutions of the system,[16–20] because solving the real solution of the system is very difficult and nearly impossible. As we all know, the qualitative analysis to the continuous coefficient system is applicable, but we always want to obtain the concrete information of the constant coefficient system. Fortunately, the method presented in this paper can deal with this problem. And this method can cope with systems with dimension more than 10.

The rest of this paper is organized as follows. Some preliminary theories about the stability of differential equations are given in Section 2. In Section 3, we provide a brief review on the numerical method for computing the real roots of a semi-algebraic system, and given a algorithm to compute the stable equilibrium. In Section 4, we

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will present the details of the stability analysis for two high-dimensional predator prey systems.

2. STABILITY ANALYSIS OF DIFFERENTIAL EQUATIONS

In this section, we review some basic theories about the stability analysis of differential equations.

Consider the following differential equations

$$\begin{cases} \dot{x}_1 = \dfrac{f_1(x)}{g_1(x)} \\ \dot{x}_2 = \dfrac{f_2(x)}{g_2(x)} \\ \vdots \\ \dot{x}_n = \dfrac{f_n(x)}{g_n(x)} \end{cases} \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n)$, $f_i(x)$, $g_i(x)$ are polynomials in polynomial ring $R[x]$, and $\dot{x}_i = dx_i/dt$ for $i = 1, 2, \ldots, n$.

A point $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ in n-dimensional real Euclidean space R^n is called an *equilibrium* of system (1) if $f_i(\bar{x}) = 0$ and $g_i(\bar{x}) \neq 0$ for $i = 1, 2, \ldots, n$.

In order to analyze the stability of an *equilibrium*, we use the famous Lyapunov's first method with the technique of linearization, by considering the following Jacobian matrix

$$J = \begin{pmatrix} \dfrac{\partial(f_1/g_1)}{\partial x_1} & \dfrac{\partial(f_1/g_1)}{\partial x_2} & \cdots & \dfrac{\partial(f_1/g_1)}{\partial x_n} \\ \dfrac{\partial(f_2/g_2)}{\partial x_1} & \dfrac{\partial(f_2/g_2)}{\partial x_2} & \cdots & \dfrac{\partial(f_2/g_2)}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial(f_n/g_n)}{\partial x_1} & \dfrac{\partial(f_n/g_n)}{\partial x_2} & \cdots & \dfrac{\partial(f_n/g_n)}{\partial x_n} \end{pmatrix} \tag{2}$$

Then the following theorem can be used to determine the stability of the *equilibrium* $\bar{x}$.

Theorem 1. *If all the eigenvalues of the matrix $J(\bar{x})$ have negative real parts then $\bar{x}$ is asymptotically stable. If $J(\bar{x})$ has at least one eigenvalues with positive real part, then $\bar{x}$ is unstable.*

For a small system, it is easy to obtain the eigenvalues of the matrix $J(\bar{x})$, then one can analysis the stability of the $\bar{x}$ using Theorem 1.

For a high-dimensional system, solving the characteristic polynomial to get the exact zeros is a difficult problem. In deed, to answer the question on stability of an *equilibrium*, we only need to know whether all the eigenvalues have negative real parts or not. Therefore, the theorem of *Routh-Hurwitz* serves to determine whether all the roots of a polynomial have negative real parts.

Theorem 2 (Hirsch and Smale, (1974)). *Routh-Hurwitz theorem: A polynomial of degree n with real coefficient*

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n, \quad a_0 > 0 \tag{3}$$

is stable if and only if the following inequality holds:

$$T_1 > 0, T_2 > 0, \ldots, T_n > 0 \tag{4}$$

where

$$T_k = \begin{pmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \ldots & 0 \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{2k-1} & a_{2k-2} & a_{2k-3} & a_{2k-4} & a_{2k-5} & a_{2k-6} & \ldots & a_k \end{pmatrix} \tag{5}$$

In Eq. (5), *if the index $j > n$, we let $a_j = 0$.*

3. ALGORITHM FOR COMPUTING THE STABLE EQUILIBRIUM

In this section, we present very briefly the method we will use for computing the stable equilibrium of the biological system.

Consider the following semi-algebraic system

$$\begin{cases} f_1 = 0, \ldots, f_n = 0 \\ p_1 > 0, \ldots, p_q > 0 \\ n_1 \geq 0, \ldots, n_s \geq 0 \\ h_1 \neq 0, \ldots, h_t \neq 0 \end{cases} \tag{6}$$

where $n \geq 1$, $q, s, t \geq 0$, We call it zero dimensional semi-algebraic system if $\{f_1, \ldots, f_n\}$ has only finite zeros in C.

Following the notations in software package Discover,[6] we let f, n, p, h denote the polynomial equations, nonnegative polynomial inequalities, positive polynomial inequalities and polynomial inequations respectively.

Our main idea is to solve the real root intervals first, and then delete some intervals which can not satisfy the restriction system by substitution and the property of interval expansion.

We first compute the solution of f using homotopy continuation method. At this step, we use the famous package homo4ps2[22] to obtain all of the approximate solutions. By using these isolated approximated solutions and the interval Newton algorithm, we obtain the isolated real root intervals of f. Now, we complete the first step to solving the real root of system (16).

Suppose we have obtained n isolated real root intervals, denoted by $X = [X_1, X_2, \ldots, X_n]$. In order to check wether each X_i fulfill those restrictions in p, we substitute interval

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X_i into p, and suppose that $p(X_i) = [I_1, I_2, \dots, I_p]$, where I_i means the interval extension of p_i at interval X_i. If there exists at least one I_j such that the right endpoint of I_j is smaller than 0, then we know that $p_j(\bar{x}) < 0$, where $\bar{x}$ is the accurate real solution of f which contained in X_i. From this, one can delete X_i from X. If all of the left endpoints of I_i, $(i = 1, 2, \dots, p)$ is bigger than 0, we have $p_i(\bar{x}) > 0$ for all $i \in \{1, 2, \dots, p\}$. At this case, the interval X_i must be retain. The most difficult case in this step is $0 \in I_i$ for some $i \in \{1, 2, \dots, p\}$, because we do not know the exact sign of p_i at $\bar{x}$, so we can not decide whether delete X_i or retain it directly. The following theorem can help us to solve this problem.

THEOREM 3 (ZHENYI JI ET AL.) *If the intersection of X_i and $\pi(Z_i)$ is an empty set for all $i \in \{1, 2, \dots, m\}$, then $p_i(\bar{x}) \geq 0$.*

where $\{Z_1, Z_2, \dots, Z_m\}$ be the isolated real root intervals of the following system

$$\tilde{f} = [f_1, f_2, \dots, f_n, a^2 p_i + 1] \tag{7}$$

and a in system (7) is a new variable. The projection π defined as follows:

$$\pi : I(\mathrm{R}^{n+1}) \to I(\mathrm{R}^n) \tag{8}$$

which remove the last coordinate of an interval vector in $I(R^{n+1})$.

Similar to theorem 3, if we construct

$$\tilde{f} = [f_1, f_2, \dots, f_n, a^2 p_i - 1] \tag{9}$$

then we can decide whether $p_i(\bar{x}) \leq 0$ or not.

Combine system (7), (8) and Theorem 3, we can get the exact sign of $p_i(\bar{x})$.

Using the method described above, we can also delete some intervals in X which don't satisfy restrictions in n, h.

So far, we complete the algorithm to get the isolated real root intervals of system (6), and named it realroot.

By combining algorithm realroot and theorem 1, we propose an algorithm to obtain the stable equilibrium of system (1).

4. STABILITY ANALYSIS OF HIGH DIMENSIONAL PREDATOR PREY SYSTEM

In this section, we use our algorithm on two high dimensional experiments. Consider the following $n+1$ dimension predator-prey system.

$$\begin{cases} \dot{x}_1 = x_1\left(a_1 - b_1 x_1 - \dfrac{cy}{d+mx_1}\right) + \displaystyle\sum_{j=1}^{n} d_{1j}(x_j - x_1), \\ \dot{x}_i = x_i(a_i - b_i x_i) + \displaystyle\sum_{j=1}^{n} d_{ij}(x_j - x_1), \quad i = 2, 3, \dots, n, \\ \dot{y} = y\left(-e + \dfrac{fx_1}{d+mx_1}\right), \end{cases} \tag{10}$$

Table I.

Algorithm 1: CSE

Input: A differential equation system in the form of (1)

1 Let polynomial system $f = [f_1, f_2, \dots, f_n]$, and $h = [g_1, g_2, \dots, g_n]$. Constructing the semi-algebraic system defined in form (6) combine f, h and the background of the biological system.

2 Run algorithm realroot to obtain the isolated real root intervals of this semi-algebraic system.

3 Let the midpoint of each interval as the approximate equilibrium, and compute the Jacobian matrix of system in right of Eq. (1) at this point.

10 **Output:** The set of stable *equilibrium* or return empty set which means that this system does't have a stable *equilibrium*.

where $x_i (i \in I = \{1, 2, \dots, n\})$ denote the population density of the prey species in the ith patch, y represents the population density of the predator species, a_i, b_i, the intrinsic growth rate and density-dependent coefficient of the prey n the ith patch, respectively. c is the capturing rate of the predator, e the death rate of the predator, f the rate of the conversion of nutrients into the reproduction of the predator and $d_{ij}(i, j \in I, i \neq j)$ he dispersal rate of he prey species from the ith patch to the jth patch. d and d are two nonnegative constants, the term $x_1/(d + mx_1)$ denotes the functional response f the predator. In this system, we always assume that the coefficients b_i, $d_{ij}(i, j \in I, i \neq j)$, c, e, and f are positive and $d_{i,i}$ for all $i \in I$.

In the following example, we let $n = 9$.

EXAMPLE 1. In this example, we give the value of each parameter in system (10) as follows:

$$c = 4.47, \quad d = 1.43, \quad m = 1.54, \quad f = 0.76$$

$$\mathbf{a} = [3.12, -0.43, 1.83, 2.79, 1.45, 0.95, 2.34, 0.45, 0.34]$$

$$\mathbf{b} = [2.43, 0.72, 1.87, 2.35, 0.64, 1.73, 1.47, 1.36, 1.62]$$

and

$$D = \begin{bmatrix} 0 & 1.32 & 0.79 & 0.45 & 0.34 & 1.23 & 1.11 & 1.36 & 0.74 \\ 2.98 & 0 & 3.12 & 3.25 & 1.67 & 1.98 & 2.61 & 3.12 & 1.79 \\ 3.1 & 1.34 & 0 & 1.45 & 1.94 & 1.04 & 2.73 & 1.39 & 4.14 \\ 2.13 & 1.74 & 1.23 & 0 & 1.46 & 2.19 & 2.32 & 1.82 & 1.27 \\ 1.24 & 1.76 & 2.76 & 2.45 & 0 & 2.67 & 2.42 & 2.53 & 1.37 \\ 3.1 & 1.34 & 1.04 & 1.45 & 1.94 & 0 & 2.73 & 3.13 & 3.84 \\ 3.2 & 1.73 & 1.73 & 2.16 & 1.47 & 1.94 & 0 & 2.43 & 3.27 \\ 3.14 & 1.65 & 2 & 3.36 & 3.27 & 2.64 & 2.98 & 0 & 3.64 \\ 1.32 & 1.73 & 1.64 & 3.3 & 3.45 & 2.97 & 2.74 & 2.1 & 0 \end{bmatrix}$$

where D_{ij} means the value of d_{ij} in system (10), and $\mathbf{a} = [a_1, a_2, \dots, a_9]$ $\mathbf{b} = [b_1, b_2, \dots, b_9]$.

First, we change this system to a polynomial system f through equating the numerators of the rational functions on the right hand of $\dot{x}_1$ and $\dot{y}$, then $h = [d + mx_1]$.

In view of the definition of the system, we have $n = []$. and $p = [x_1, x_2, \dots, x_9, y]$.

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Indeed, polynomial system f has four real root. We obtain the following four isolated real root intervals before substitute these interval into constraints.

$$X_1 = \begin{array}{l} [0.810497781914894, 0.810497782402504] \\ [0.804586919132605,\ 0.804586919605285] \\ [0.848395109825681,\ 0.848395110322056] \\ [0.879296392344588,\ 0.879296392853449] \\ [0.883548279519186,\ 0.883548280034860] \\ [0.816732188901196,\ 0.816732189381768] \\ [0.881105623407286,\ 0.881105623920952] \\ [0.820852875050071,\ 0.820852875531826] \\ [0.809387608678938,\ 0.809387609153951] \\ [0.820326466987962,\ 0.820326473136347]. \end{array}$$

$$X_2 = \begin{array}{l} [-1.000000000000000, 1.000000000000000] \\ [-1.000000000000000,\ 1.000000000000000] \\ [-1.000000000000000,\ 1.000000000000000] \\ [-1.000000000000000,\ 1.000000000000000] \\ [-1.000000000000000,\ 1.000000000000000] \\ [-1.000000000000000,\ 1.000000000000000] \\ [-1.000000000000000,\ 1.000000000000000] \\ [-1.000000000000000,\ 1.000000000000000] \\ [-1.000000000000000,\ 1.000000000000000] \\ [-1.000000000000000,\ 1.000000000000000] \end{array}$$

$$X_3 = \begin{array}{l} [1.075972957834162, 1.075972959460317] \\ [0.972173322154557,\ 0.972173323254987] \\ [1.016351106633443,\ 1.016351107746212] \\ [1.042469515727628,\ 1.042469516821479] \\ [1.059470463111973,\ 1.059470464274978] \\ [0.980875353753226,\ 0.980875354839560] \\ [1.058542611101414,\ 1.058542612270612] \\ [0.987468461441554,\ 0.987468462540937] \\ [0.964765762461909,\ 0.964765763500400] \\ [-0.000000000753366,\ 0.000000000753366] \end{array}$$

$$X_4 = \begin{array}{l} [0.810497782139752, 0.810497782177646] \\ [-1.086117396750171,\ -1.086117396597172] \\ [-1.277719467538150,\ -1.277719467314553] \\ [-1.917258329539363,\ -1.917258329253069] \\ [-1.360235897918054,\ -1.360235897735432] \\ [-1.169995339202457,\ -1.169995339019288] \\ [-1.267061553518377,\ -1.267061553329044] \\ [-1.183977886696930,\ -1.183977886523332] \\ [-1.404074548275796,\ -1.404074548072315] \\ [-10.555921198085869,\ -10.555921193470624] \end{array}$$

We delete X_2, X_3, X_4 using algorithm realroot. This shows that there are four equilibrium of the system, only one of them is positive. Let the midpoint of X_1 be the approximate equilibrium, and compute the eigenvalues of the Jacobian matrix at this point, we obtain the following ten eigenvalues,

$$\begin{array}{l}[-1.26707938086,\ -.0436116879983,\ -11.0076355860, \\ -17.579017256,\ -25.4088078712,\ -21.832928144, \\ -23.805150198-.441343035292i,\ -20.2687098484 \\ +.410637687497i,\ -20.26870985-.41063768749i, \\ -23.80515019+.44134303529i]\end{array}$$

Then we know that this point is a stable equilibrium based on Theorem 2, which guarantees the predator-prey system described by Eq. (10) is permanent and has only one positive stable state.

EXAMPLE 2. In this example, we let $n = 10$, and the values of parameter are mentioned as:

$$c = 4.76, \quad d = 2.53, \quad m = 1.73, \quad f = 2.43,$$

$$\mathbf{a} = [0.32, 2.43, 0.54, 3.21, 3.23, 1.95, 2.34, 0.45, 0.34, 0.74]$$

$$\mathbf{b} = [3.34, 1.32, 2.68, 2.53, 1.34, 2.37, 1.47, 1.36, 1.62, 1.17]$$

$$D = \begin{bmatrix} 0 & 1.24 & 1.76 & 0.45 & 2.69 & 0.96 & 1.37 & 0.24 & 1.54 & 2.65 \\ 1.23 & 0 & 1.23 & 2.53 & 2.64 & 0.87 & 1.64 & 2.34 & 3.56 & 4.23 \\ 1.23 & 4.35 & 0 & 3.64 & 2.15 & 2.56 & 3.54 & 4.52 & 1.34 & 3.52 \\ 3.15 & 2.53 & 3.46 & 0 & 2.46 & 1.47 & 2.46 & 1.26 & 2.16 & 3.14 \\ 3.21 & 2.45 & 4.32 & 3.45 & 0 & 3.74 & 1.34 & 4.56 & 3.24 & 2.45 \\ 2.38 & 2.86 & 3.45 & 3.46 & 4.65 & 0 & 2.34 & 2.17 & 2.79 & 3.65 \\ 3.2 & 1.73 & 1.73 & 2.16 & 1.47 & 1.94 & 0 & 2.43 & 3.27 & 3.46 \\ 3.14 & 1.65 & 2 & 3.36 & 3.27 & 2.64 & 2.98 & 0 & 3.64 & 4.53 \\ 1.32 & 1.73 & 1.64 & 3.3 & 3.45 & 2.97 & 2.74 & 2.1 & 0 & 3.54 \\ 0.34 & 0.27 & 1.23 & 2.63 & 1.94 & 1.79 & 1.63 & 1.32 & 1.24 & 0 \end{bmatrix}$$

Like Example 1, the system also have four real roots, and we delete three of them through the constraints. Using theorem 2, we prove that the point

$$\begin{array}{l}[0.64521677243342646,\ 0.45062858327973992, \\ 0.79108902305373541,\ 0.72702509245488045, \\ 0.78874125346439294,\ 0.79943477664536378, \\ 0.76117242763987458,\ 0.78554440360188804, \\ 0.73896454660658983,\ 0.73664609495876254, \\ 0.75661214439091529]\end{array}$$

is the only one stable equilibrium of the differential system.

5. CONCLUSIONS

For a high dimensional predator prey system, based on the numerical method for isolation the real roots of semi-algebraic system, we compute the equilibrium first, and determine its stability based on the Lyapunov's first method. Using our method, for two predator prey systems with dimension ten and eleven respectively, all of the equilibrium are found, and we confirm that there are only one positive stable equilibrium.

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