ON A QUESTION OF CSIMA ON COMPUTATION-TIME DOMINATION

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1. Introduction

The settling-time reducibility, $<_{st}$, among the c.e. sets, was first proposed by Nabutovsky and Weinberger in [5] and Soare in [6]. For a computably enumerable (*c.e.* for short) set A with an effective enumeration $\{A_s\}_{s\in\omega}$, the *settling function* of A w.r.t. this enumeration is

$$m_A = \mu s[A_s \upharpoonright x = A \upharpoonright x],$$

where $A \upharpoonright x = \{y \leq x |\ y \in A\}$.

Definition 1.1. Let A and B be two *c.e.* sets, with effective enumerations $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$ respectively. Say that A settling-time dominates B, denoted as $B <_{st} A$ if for any computable function f, $m_A(x) > f(m_B(x))$ is true for almost all x, i.e. m_A dominates $f \circ m_B$.

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Nies observed that this reduction does not depend on the choice of enumerations, and hence $<_{st}$ is an ordering on *c.e.* sets. In [2], Csima considered a direct generalization of $<_{st}$ to Δ^0_2 sets.

Definition 1.2. Let A be a Δ^0_2 set with effective approximation $\{A_s\}_{s\in\omega}$. The *settling function* for A w.r.t. this enumeration, $m_A(x)$, is defined as the least stage after which the approximation does not change up to x:

$$m_A(x) = (\mu s)(\forall t \geq s)[A_t \upharpoonright x = A \upharpoonright x].$$

Csima proved that $<_{st}$ on Δ^0_2 set is not reflexive, as for each $n \geq 2$, there is a properly n-*c.e.* set A with two n-*c.e.* approximations $\{A_s\}_{s\in\omega}$ and $\{\widetilde{A}_s\}_{s\in\omega}$ such that for any total computable function f, $m_{\tilde{A}}$ dominates $f(m_A)$. When $\{A_s\}_{s\in\omega}$ is an effective enumeration of a given Δ^0_2 set, the settling function of $\{A_s\}_{s\in\omega}$ is quite different from the computation function, even though they coincide when the given set is c.e.

Definition 1.3. Let A be a Δ^0_2 set with effective approximation $\{A_s\}_{s\in\omega}$. The *computation function* of A w.r.t. this enumeration, $C_A(x)$, is the least stage s such that A_s and A agree up to x. That is,

$$C_A(x) = (\mu s \geq x)[A_s \upharpoonright x = A \upharpoonright x].$$

It is a folklore that for any Δ^0_2 set A, $C_A \equiv_T A$. Based on Nabutovsky and Weinberger's notion of settling-time dominating on c.e. sets, Csima proposed the notion of computation-time dominating among Δ^0_2 sets.

Definition 1.4. Let A and B be two Δ^0_2 sets, with effective approximation $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$ respectively. Say that A computation-time dominates B w.r.t. to these enumerations, if for all total computable function f, $C_A(x) > f(C_B(x))$ is true for almost all x, i.e. C_A dominates $f \circ C_B$.

In [2], Csima showed that Nies' observation that settling-time dominating does not depend on the enumerations is not true for computation-time dominating, when Δ^0_2 enumerations are involved. What Csima proved is the existence of c.e. set A with a c.e. enumeration $\{A_s\}_{s\in\omega}$ and a 3-c.e. approximation $\{\tilde{A}_s\}_{s\in\omega}$ such that the computation function of $\{A_s\}_{s\in\omega}$ dominates $f \circ C_{\tilde{A}}$ for any total computable function f. In the paper, Csima asked whether this kind of computation-time domination ordering depending on approximations is also true for 2-c.e. sets. In this paper, we give a confirmative answer to Csima's question.

Theorem 1.1. *There exists a 2-c.e. set A with two different 2-c.e. approximations $\{A_s\}_{s\in\omega}$ and $\{\widetilde{A}_s\}_{s\in\omega}$ such that for every total computable function f, such that for almost all x,*

$$C_{\widetilde{A}}(x) \geq f(C_A(x)).$$

Our notations and terminology are standard and generally follow Soare [7].

2. Requirements and basic strategy

We will construct a 2-*c.e.* set A with two different 2-*c.e.* approximations $\{A_s\}_{s\in\omega}$ and $\left\{\widetilde{A}_s\right\}_{s\in\omega}$ such that for each e, the following requirements are satisfied:

$$\mathcal{R}_e : \varphi_e \text{ is total } \Rightarrow (\forall^\infty x)[C_{\widetilde{A}}(x) \geq \varphi_e(C_A(x))],$$

where φ_e is the e^{th} partial computable function.

Csima's idea of satisfying a single $\mathcal{R}_e$ requirement is direct. That is, first partition ω into infinitely many nonempty blocks, B_n, $n \in \omega$. To satisfy $\mathcal{R}_e$, it is enough to show that for each $n \geq e$, for all numbers x in B_n, $C_{\widetilde{A}}(x) \geq \varphi_e(C_A(x))$. Here is the idea. Let x_n be an element in B_n, and put x_n into A_s at stage s, remove it at stage $s+1$. According to [2], this is called a tempting process. Now wait for $\varphi_e(s)$ to converge. If it does not converge, then $\mathcal{R}_e$ is satisfied. Otherwise, let $t > s$ be a stage at which $\varphi_e(s)$ converges, and we do at stage t is to reenumerate x_n into A_t, and also into $\widetilde{A}_t$. Note that $A_t \restriction (x_n+1) = A_s \restriction (x_n+1)$, and so $C_A(x_n+1) = s$. By $\varphi_e(s) < t$, we have

$$C_{\widetilde{A}}(x_n+1) = t > \varphi_e(s) = \varphi_e(C_A(x_n+1)).$$

Csima's idea works well to show that for all numbers x in B_n, $C_{\widetilde{A}}(x) \geq \varphi_e(C_A(x))$ and hence $\mathcal{R}_e$ is satisfied. In Csima's argument, $\cup\widetilde{A}_s$ is a c.e. set, while $\{A_s : s \in \omega\}$ is a 3-c.e. approximation of this set. It seems that the use of one element for temptation of each block makes a 3-c.e. approximation quite necessary.

Our construction is a variant of Csima's original idea, and is able to make approximations 2-c.e. Instead of using one element for the temptation, we use two-element. That is, we start the temptation by enumerating a number $x+1$ into A_s, and then enumerating x at stage $s+1$. At stage $s+1$, we also enumerate both x and $x+1$ into $\widetilde{A}_{s+1}$ (we do this to ensure that $\lim_s A_s$ and $\lim_s \widetilde{A}_s$ are equal). Now if $\varphi_e(s)$ converges at stage t, then $\varphi_e(s) < t$, and we remove x from both A_t and $\widetilde{A}_t$. Then $A_t \upharpoonright (x+2) = A_s \upharpoonright (x+2)$ and $C_A(x+2) = s$. As $C_{\widetilde{A}}(x+2) = t$, $C_{\widetilde{A}}(x) \geq \varphi_e(C_A(x))$. Again, we can show that $\mathcal{R}_e$ is satisfied.

We now consider the general case. As in [2], we partition ω into infinitely many consecutive blocks B_0, B_1, $\cdots$, such that each block B_n is composed of n^3+1 subblocks, $B_{n,0}$, $B_{n,1}$, $\cdots$, B_{n,n^3}, and each subblock contains $2n+2$ numbers. Here the numbers of $B_{n,i}$ are less than the numbers in $B_{n,j}$, if i is less than j, and we will make sure that for each $e < n$, if φ_e is total, then $C_{\widetilde{A}}(x) \geq \varphi_e(C_A(x))$ is true for all x in B_n. It can be done on a single subblock, and we let B_n have n^3+1 subblocks because whenever we do actions on some block B_m with $m < n$, we need to restart our work on B_n on a new subblock. We will see that n^3+1 blocks is enough to deal with the actions of these B_m's.

Say that we activate block B_n at stage s via an unused subblock $B_{n,k} = \{b_0, b_1, \cdots, b_{2n+1}\}$ with $b_0 < b_1 < \cdots < b_{2n+1}$, if $B_{n,k}$ is the largest unused subblock of B_n, and our actions at stage s and the following stages are:

- at stage s, enumerate $b_{2n+1}, \cdots, b_3, b_1$, into A_s;
- at stage $s+1$, enumerate b_{2n} into A_{s+1}
- at stage $s+2$, enumerate b_{2n-2} into A_{s+2},

  $\cdots$
- at stage $s+n+1$, enumerate b_0 into A_{s+n+1}.

  We use c_n to denote $s+n+1$, for convenience.

Following [2], we call this a *temptation* process. We say that B_n is fully activated at stage $s+n+1$ via subblock $B_{n,k}$ if this process is done, and from this stage, B_n continues to be active, until we do actions on some block $B_m, m < n$ (if so, we say that block B_n is deactivated, and consequently, c_n is undefined automatically), and B_n needs to be activated later via a(nother) unused subblock (a new c_n will be defined in this case), which means that we abandon the subblock $B_{n,k}$ used just now, and will activate B_n via $B_{n,k-1}$. In the construction, at any stage, for each block B_n, at most one subblock is active, and the first subblock of B_n being activated will be the greatest one, i.e. B_{n,n^3}. A subblock $B_{n,k}, k < n^3$, will be activated only after $B_{n,k+1}$, the previously activated subblock, is deactivated by some action on a smaller block.

In the construction, the temptation process can skip several stages, so when we say that a computation becomes convergent at stage s, we mean this computation has not been convergent previously, except those stages being skipped in some temptation process.

We say that a block B_n requires attention at stage s if one of the following occurs:

- B_n is not active (so c_n is undefined), or
- There is some $e \leq n$ such that $\varphi_e(c_n)$ becomes convergent at stage s.

3. Construction

We now provide the construction of the 2-c.e. set, and two different 2-c.e. approximations. First, we partition ω into infinitely many blocks $B_n, n \in \omega$, such that each block B_n is composed of $n^3 + 1$ subblocks, $B_{n,0}$, $B_{n,1}$, $\cdots$, B_{n,n^3}, and each subblock contains $2n + 2$ numbers.

Stage 0: Do nothing.

Stage $s > 0$: Activate block B_s first via the greatest subblock $B_{n,n\cdot 2^n} = \{b_0, b_1, \cdots, b_{2n+1}\}$ and do the temptation process as follows:

- at stage s, enumerate $b_{2s+1}, \cdots, b_3, b_1$, into A_s;
- at stage $s + 1$, enumerate b_{2s} into A_{s+1}
- at stage $s + 2$, enumerate b_{2s-2} into A_{s+2},
  $\ldots$
- at stage $2s + 1$, enumerate b_0 into A_{2s+1}; and enumerate $b_0, b_1, \cdots, b_{2n+1}$ into $\widetilde{A}_{2s+1}$. Set $c_n = 2s + 1$.

Now we consider whether there are some n and $e \leq n$ such that B_n is active via a subblock $B_{n,k}$ and $\varphi_e(c_n)$ becomes convergent at stage s. Find the largest $i \leq n$ such that $b_{2i} \in A_s \cap B_{n,k}$, and remove b_{2i} from both A and $\widetilde{A}$. We say that we do actions on B_n at stage s. For $m > n$, if B_m is active via a subblock $B_{m,l}$, deactivate $B_{m,l}$, do the temptation process for subblock $B_{m,l-1}$, and declare that B_m is active via a subblock $B_{m,l-1}$. Redefine $c_m = s + m + 1$.

Go to stage $2s + 2$. *We skip the stages from $s + 1$ to $2s + 1$ as we do the temptation during these stages.*

This completes the construction.

4. Verification

We now verify that the constructed approximations $\{A_s\}_{s\in\omega}$ and $\{\widetilde{A}_s\}_{s\in\omega}$ satisfying the requirements. Obviously, $\lim_s A_s = \lim_s \widetilde{A}_s$, and both $\{A_s\}_{s\in\omega}$ and $\{\widetilde{A}_s\}_{s\in\omega}$ are 2-*c.e.* approximations.

Now we check that each block B_n has enough subblocks to carry out the temptations.

Lemma 4.1. *For each block B_n, $n \in \omega$, there is a subblock $B_{n,k}$ and a stage s such that after stage s, B_n keeps active via $B_{n,k}$ forever.*

Proof. By the construction, the active subblock of B_n can be shifted to a smaller one only when some number in B_m, $m < n$, is removed from A and $\widetilde{A}$. So this kind of shifting can happen at most

$$1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdots 2^n < n \cdot 2^n$$

many times, and $n \cdot 2^n$ many subblocks in B_n enables us to find an unused subblock in B_n whenever needed. As a consequence, we will activate a subblock $B_{n,k}$ of B_n at some stage in the construction, and from then on B_n will not be deactivated afterwards. That is, B_n keeps active via subblock $B_{n,k}$ forever. □

The following lemma shows that each requirement is satisfied.

Lemma 4.2. *For each $e \in \omega$, if φ_e is total, then for each $n \geq e$, and for all $x \in \cup_{n\geq e} B_n$, $C_{\widetilde{A}}(x) \geq \varphi_e(C_A(x))$. $\mathcal{R}_e$ is satisfied.*

Proof. First we prove by induction on n that we do actions on each B_n at most finitely often. Suppose that it is true for B_j, $j < n$. Let s_n be the last stage we do actions for these B_js. Then after stage s_n, after the temptation for B_n (the last one) has been done, as c_n is now fixed, we do actions on block B_n, because $\varphi_e(c_n)$ becomes convergent for some $e \leq n$. Obviously, we do such actions on block B_k at most $n + 1$ many times. It completes the induction.

Now we show that for each e, if $n \geq e$, then for all $x \in B_n$, $C_{\widetilde{A}}(x) \geq \varphi_e(C_A(x))$. By the statement above, let s, k be the numbers such that after stage s, $B_{n,k}$ keeps active forever. This means that c_n becomes fixed.

Let y be the least number in $B_{n,k}$ being removed from A and $\widetilde{A}$. Obviously, y is the last number being removed from block B_n, and hence for

$x \geq y, x \in B_n$, $C_{\widetilde{A}}(x) = C_{\widetilde{A}}(y) = t$. Suppose that y is removed at stage t. As $C_A(x) \leq c_n$, if φ_e, $e < n$, is total, then $\varphi_e(c_n)$ converges by stage t; otherwise, when it converges later, we have to do action on B_n again, contradicting our assumption on y. Therefore $\varphi_e(C_A(x)) \downarrow [t] < t = C_{\widetilde{A}}(x)$ by the convention that $\varphi_e(n) \downarrow [t]$ implies $\varphi_e(m) \downarrow [t]$ for all $m < n$.

Now, we consider $x < y$ in B_n. Because a fresh subblock of B_n is always activated when we remove number from B_{n-1}. It follows that $C_{\tilde{A}}(x) = C_{\tilde{A}}(\max B_{n-1})$, $C_A(x) = C_A(\max B_{n-1})$ for all $x \in B_n$ less than y. Since $e < n$, φ_e would have been considered in the block B_{n-1}. Using the previous argument, we have $C_{\tilde{A}}(\max B_{n-1}) > \varphi_e(C_A(\max B_{n-1}))$. Therefore, $C_{\tilde{A}}(x) > \varphi_e(C_A(x))$. □

This complete the proof of Theorem 1.1.

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